Homomorphisms

In mathematics it is important to talk about maps with "preserve structure".
In linear algebra, these are the linear transformations.
In set theory, this is any function.

It is also important to talk about maps which "embed" one object into another.
In linear algebra, these are the injective transformations.
In set theory, these are the injective functions.

Last it is important to talk about when two objects are "exact copies" of each other.
In linear algebra, this is isomorphic spaces.
In set theory, this is equinumerous sets.
**Def.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be structures for the language. A **homomorphism** \( h \) of \( \mathcal{A} \) into \( \mathcal{B} \) is a function \( h: |\mathcal{A}| \rightarrow |\mathcal{B}| \) such that

1. for any \( n \)-place predicate parameter \( P \) and each \( n \)-tuple \( \langle a_1, \ldots, a_n \rangle \) of elements of \( |\mathcal{A}|_1 \),
   \[ \langle a_1, \ldots, a_n \rangle \in P_{\mathcal{A}} \iff \langle h(a_1), \ldots, h(a_n) \rangle \in P_{\mathcal{B}} \]

2. for any \( n \)-place function symbol \( f \) and each such \( n \)-tuple,
   \[ h(f^\mathcal{A}(a_1, \ldots, a_n)) = f^\mathcal{B}(h(a_1), \ldots, h(a_n)) \]
   For constants \( c \), this becomes
   \[ h(c^\mathcal{A}) = c^\mathcal{B} \].

**Def.** If \( h \) is one-to-one, then say \( h \) is an **isomorphism** of \( \mathcal{A} \) into \( \mathcal{B} \) or \( h \) is an **isomorphic embedding** of \( \mathcal{A} \) into \( \mathcal{B} \).

If \( h \) is one-to-one and onto, then say \( h \) is an **isomorphism** of \( \mathcal{A} \) onto \( \mathcal{B} \).

Also say \( \mathcal{A} \) and \( \mathcal{B} \) are **isomorphic**.
Examples

1. Language: \( \mathbb{A}, +, \cdot \)
   \( \mathbb{A} = (\mathbb{N}, +, \cdot) \)
   \( h: \mathbb{N} \to \{e, o\} \) defined by

   \[ h(n) = \begin{cases} e & \text{if } n \text{ is even} \\ o & \text{if } n \text{ is odd} \end{cases} \]

   This defines an 'onto homomorphism' from \( \mathbb{A} \) onto \( \mathbb{B} \), where \( |\mathbb{B}| = \{e, o\} \) and 
   \(+, \cdot\) are given by the tables

   \begin{align*}
   + & \quad e \quad o \\
   e & \quad e \quad e \\
o & \quad e \quad o
   \end{align*}

   \begin{align*}
   \cdot & \quad e \quad o \\
e & \quad e \quad e \\
o & \quad o \quad e
   \end{align*}

   Verification: If \( a, b \) odd then \( a + b \) even, so
   \[ h(a + b) = e \]
   \[ h(a) + h(b) = e \]
   \( (a + b) \cdot o = e \)

   (can check all other cases)
2. $P = \text{positive integers } = \{1, 2, \ldots, 3\}$
   $\prec_P = \text{ordering relation on } P$

3. $\prec_N = \text{ordering relation on } IN$

4. There is an isomorphism $h$ from $(P, \prec_P)$ onto $(N, \prec_N)$:
   $h(n) = n - 1$

5. There is an isomorphism from $(P, \prec_P)$ into $(N, \prec_N)$:
   $\text{id}: P \rightarrow N$

Prop. If $A$ and $B$ are structures for a language such that $|A| \subseteq |B|$. Then
the identity map $\text{id}: |A| \rightarrow |B|$ is an isomorphism if

- $P^A$ equals $P^B$ restricted to $|A|$ for all predicates
- $f^A$ equals $f^B$ restricted to $|A|$ for all functions
- $c^A = c^B$ for all constants

Def. If the above conditions are met, say $A$ is a substructure of $B$ and
$B$ is an extension of $A$.

Examples:
1. $(\mathbb{Q}; +, \cdot, 0)$ is a substructure of $(\mathbb{C}; +, \cdot, 0)$
2. Every subset of $(\mathbb{N}; <)$ is a substructure (nonempty)