Last time

Thm 22A  \( s_1 : V \rightarrow 1 \mathbb{Q} \) \( s_2 : V \rightarrow 1 \mathbb{Q} \)
agree on all variables occurring free in \( \varphi \), then
\[ l = \varphi [s_3] \iff l = \varphi [s_2] \]

Therefore we can think of each formula with free variables among \( V_1, \ldots, V_n \) as inducing a function \( 1 \mathbb{Q}^n \rightarrow \{ F, T \} \) where it returns

\[ \begin{cases} T & \text{if } l = \varphi [\alpha_1, \ldots, \alpha_n] \\ F & \text{if } l \neq \varphi [\alpha_1, \ldots, \alpha_n] \end{cases} \]

Example \( \forall \mathbb{Q}^n \)

\( (N : j \leq s, S, 0) \)

\[ l = \forall V_2, P V_1 V_2 [0] \]
\[ \neq \forall V_2, P V_1 V_2 [5] \]

Recall

\[ l = \varphi [\alpha_1, \ldots, \alpha_n] \]
\[ \neq \varphi [s_3] \]
for some (or all)
\( s : V \rightarrow 1 \mathbb{Q} \) s.t.
\( s(v_i) = a_i \quad (i \leq n) \)
Corollary 220. For a sentence $\sigma$, either

- $\mathcal{A}$ satisfies $\sigma$ with every function $s : V \rightarrow |\mathcal{A}|$.
- $\mathcal{A}$ does not satisfy $\sigma$ with any such function.

(i.e. it is either all $s$ or no $s$)

Def 12. (a) holds, then say that

- $\sigma$ is true in $\mathcal{A}$
- $\mathcal{A} \models \sigma$
- $\mathcal{A}$ is a model of $\sigma$.

Also $\mathcal{A}$ is a model of a set of sentences $\Sigma$ if $\mathcal{A}$ is a model of each $\sigma \in \Sigma$. 

Examples

(A) \( R = (R; 0, 1, +, x) \)
\( Q = (Q; 0, 1, +, x) \)

\( \exists x \ (x \cdot x = 1+1) \) is true in \( R \)

is not true in \( Q \)

(8) Parameters: \( A, P \) (2-place predicate)
\( \mathcal{A} = (\{A\}; P^a) = (A; R) \)

\uparrow \text{set} \uparrow \text{binary relation}

Classes of models:
\( (A; R) \) is a model of...

1. \( \forall x \forall y \ x = y \)
   iff \( A \) has exactly one element, \( A = \{a\} \)
   \( R \) doesn't matter, so we have two choices
   \( R = \emptyset \) or \( R = \{<a, a>\} \)

2. \( \forall x \forall y \ P_{xy} \)
   iff \( A \) is any nonempty set and \( R = A \times A \)

3. \( \forall x \forall y \neg P_{xy} \)
   iff \( A \) is any nonempty set and \( R = \emptyset \)

4. \( \forall x \exists y \ P_{xy} \)
   iff \( A \) is any nonempty set and \( \text{dom} R = A \)

5. One can also find sentences \( \exists a: \exists b \ (a, b) \in R \)
   saying \( \{a\} \) has exactly six elements,
   or saying \( P^a \) defines a function.
Logical implication

In sentential logic we used \( \models \) to mean "tautological implication". Here we will use \( \models \) to mean something slightly different.

**Def.** Let \( \Gamma \) be a set of well-formed formulas (wffs).
Let \( \varphi \) be a wff.
 Say \( \Gamma \) logically implies \( \varphi \) \( (\Gamma \models \varphi) \)
iff for every structure \( \mathcal{A} \) for the language and every function \( s: \mathcal{V} \rightarrow \mathcal{A} \),
if \( \mathcal{A} \) satisfies every member of \( \Gamma \) with \( s \),
then \( \mathcal{A} \) also satisfies \( \varphi \) with \( s \).

**Notation.** \( \models \varphi \) instead of \( \vDash_{\mathcal{A}} \models \varphi \)
\( \models \varphi \) instead of \( \vDash_{\mathcal{A}} \models \varphi \)

\( \varphi \) is valid iff \( \models \varphi \).

\( \varphi \) and \( \psi \) are logically equivalent iff \( \models \varphi = \models \psi \)
iff \( \models \varphi = \models \psi \) and \( \models \psi = \models \varphi \).

**Cor 22C**
For a set \( \Sigma \) of sentences, 
\( \Sigma \models \varphi \) iff every model of \( \Sigma \) is also a model of \( \varphi \). A sentence \( \varphi \) is valid iff it is true in every structure.