Compactness

**Def:** A set of wffs is **satisfiable** if there is a truth assignment that satisfies every member of \( \Sigma \).

A set of wffs \( \Sigma \) is **finitely satisfiable** if every finite subset \( \Sigma_0 \subseteq \Sigma \) is satisfiable.

**Compactness Theorem**

A set of wffs is satisfiable if it is finitely satisfiable.

**Proof**

One direction (the "only if" direction) is easy.

For the other direction:

1. Extend a finitely satisfiable \( \Sigma \) to a maximal finitely satisfiable \( \Delta \).
2. Use \( \Delta \) to find a truth assignment satisfying \( \Sigma \).
Let $\alpha_1, \alpha_2, \ldots$ be a fixed enumeration of the wffs. (This can be done since there are countably many symbols. Therefore for each length $n$, there are countably many expressions of length $n$. Then the set of expressions is $\bigcup_n A_n$ where $A_n$ is the set of expressions of length $n$. A countable union of countable sets is countable, hence a subset of a countable set is countable.

... better yet, just systematically write down all the wffs.)

Define by recursion (on $\mathbb{N}$)

$$\begin{align*}
\Delta_0 &= \Sigma \\
\Delta_{n+1} &= \left\{ \begin{array}{ll}
\Delta_n & \text{if this is finitely satisfiable} \\
\Delta_n \cup \alpha_{n+1} & \text{otherwise}
\end{array} \right.
\end{align*}$$

Claim: Each $\Delta_n$ is finitely satisfiable.
Proof: HW exercise. (#1, p.65)

Let $\Delta = \bigcup \Delta_n$.

Three properties of $\Delta$:
1. $\Sigma \subseteq \Delta$
2. For all wff $\alpha$, $\alpha \in \Delta$ or $(\neg \alpha) \in \Delta$.
3. $\Delta$ is finitely satisfiable.
Part 2

Let \( u \) be the truth assignment given by

\[
u(A_n) = T \quad \text{iff} \quad A_n \in \Delta
\]

for all sentence symbols \( A_n \). (Note \( A_n \) is a well-formed formula. Either \( A_n \in \Delta \) or \( \neg A_n \in \Delta \).

Claim: For all well-formed formulas \( \varphi \),

\( u \) satisfies \( \varphi \) iff \( \varphi \in \Delta \).

Proof: HW problem (use induction) (#2, p 65)

Since \( \Sigma \subseteq \Delta \), \( u \) satisfies every member of \( \Sigma \).
**Corollary 17.4**

If \( \Sigma \models \phi \), then there is a finite \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \models \phi \).

**Proof**

*Easy fact* \( \Sigma \models \phi \) iff \( \Sigma \setminus \phi \) is unsatisfiable.

Then

\[
\begin{align*}
\Sigma_0 \not\models \phi & \text{ for all finite } \Sigma_0 \subseteq \Sigma \\
\Rightarrow \Sigma_0 \setminus \phi & \text{ is satisfiable for all finite } \Sigma_0 \subseteq \Sigma \\
\Rightarrow \Sigma \setminus \phi & \text{ is finitely satisfiable} \\
\Rightarrow \Sigma & \text{ is satisfiable (compactness theorem)} \\
\Rightarrow \Sigma & \not\models \phi.
\end{align*}
\]

(This is equivalent to compactness theorem.)

HW exercise, #3 p.68
Effectiveness

Question
Given $\exists \tau$ can we "decide" whether $\exists \tau$ using an "effective procedure"?

Informally, an effective procedure must meet the following conditions:

1. It is given by a finite set of instructions. (Think computer program or instructions for a human assistant who doesn't know math.)

2. They must be mechanically implemented (no intelligence is assumed on the machine or person following the instructions. Also, no randomness (flipping a coin) may be used.

3. After a finite number of steps it produces yes/no.

Notes: It has no upper bound on finite. The instructions may be as large as needed, the time may be as long as needed, and the resources may be as much as needed.
**Theorem 17B**

There is an effective procedure for deciding whether an expression is a wff.

**Proof** See algorithm in Section 1.3 □

**Remark** To overcome issues of having infinitely many symbols, we can use $A'''$ in place of, say, $A_4$. Then there are only 10 symbols (, ), $\forall, \land, \lor, \Rightarrow, \equiv, A_j$ which we can identify with 0–9 if we like.

**Def** A set $\Sigma$ of expressions is decidable iff there is an effective procedure that, given an expression $\alpha$, will decide whether or not $\alpha \in \Sigma$.

**Thm 17C**

There is an effective procedure that, given a finite set $\Sigma; \varepsilon$ of wffs, will decide whether or not $\Sigma \varepsilon \varepsilon$.

**Proof** Use a truth table. □

**Corollary** The set of tautologically consequences of $\Sigma$ is decidable for finite $\Sigma$. (This includes tautologies.)