Recursion:

It is important to be able to define a function recursively:

Assume \( S \) is a set generated from \( B \) by \( f \) and \( g \).
(This is just a special case.)
Define \( \overline{h} : S \rightarrow S \) as follows:
1. Define \( \overline{h}(x) = h(x) \) for \( x \in B \)
2. Define \( \overline{h}(g(x,y)) \) by using \( \overline{h}(x) \) and \( \overline{h}(y) \) only.
3. Define \( \overline{h}(g(x)) \) by only using \( \overline{h}(x) \).

Good examples:

1. \( \overline{h} : \mathbb{N} \rightarrow \mathbb{N} \)
   \( s(x) = x + 1 \)
   \( \overline{h}(0) = 1 \)
   \( \overline{h}(s(x)) = \begin{cases} 0 & \text{if } \overline{h}(x) = 1 \\ 2 & \text{if } \overline{h}(x) = 0 \end{cases} \)

2. \( \overline{h} : \mathbb{N} \rightarrow \mathbb{N} \)
   \( s(x) = x + 1 \)
   \( \overline{h}(0) = 1 \)
   \( \overline{h}(s(x)) = \overline{h}(x) \cdot 2 \)

3. \( \overline{h} \) defined on \( \mathcal{W} \)
   \( \overline{h}(A) = 1 \)
   \( \overline{h}(\neg \phi) = \overline{h}(\phi) \)
   \( \overline{h}(\alpha \lor \beta) = \overline{h}(\alpha) + \overline{h}(\beta) \) for \( \alpha \in \{ \land, \lor, \rightarrow, \leftrightarrow \} \)
Bad example

\[ f : \mathbb{Z} \to \mathbb{Z} \]
\[ f(0) = 1 \]
\[ f(S(x)) = f(x) + 2 \]
\[ f(P(x)) = f(x) \cdot 2 \]

Then
\[ f(0) = 1 \]
\[ f(0) = f(P(S(0))) \]
\[ = f(S(0)) \cdot 2 \]
\[ = (f(0) + 2) \cdot 2 \]
\[ = (1 + 2) \cdot 2 = 6 \neq 1 \]

The issue is that there are multiple paths to get to 0.

However, for \( \mathbb{N} \) and the wffs, there is only one path.
Def Let $C$ be generated from $B$ by $f$ and $g$. Let $f_C$ and $g_C$ be the restrictions of $f$ and $g$ to $C$. We say $C$ is freely generated from $B$ by $f$ and $g$ if

1. $f_C$ and $g_C$ are one-to-one
2. The range of $f_C$ and the range of $g_C$, and the set $B$ are pairwise disjoint.

Rmk The term "free" comes from algebra, as in "the free group on two elements".
Recursion Theorem

Assume $C \subseteq U$ and $C$ is \underline{freely} generated from $B$ by $\bar{F}$ and $g$, where

$\bar{F} : U \times U \rightarrow U$

$g : U \rightarrow U$

Assume $V$ is a set and there are functions

$h : B \rightarrow V$

$F : V \times V \rightarrow V$

$G : V \rightarrow V$

Then $h$ can be uniquely extended to

$\bar{h} : C \rightarrow V$

where

1. For all $x \in B$, $\bar{h}(x) = h(x)$
2. For all $x, y \in C$,
   
   $\bar{h}(\bar{F}(x, y)) = F(\bar{h}(x), \bar{h}(y))$
   
   $\bar{h}(g(x)) = G(\bar{h}(x))$.

Remark: Again, $\bar{F}$ and $g$ are just examples. This theorem holds for all sets of functions $\bar{F}$.

Remark: This theorem gives a recipe for "painting" all the elements of $C$. The "freely" lets one avoid painting the same element more than once.
Examples of freely generated sets

1. \( \mathbb{N} \) is freely generated from \( \mathbb{I} \) by \( S(x) = x + 1 \).

   **Proof:** \( S \) has a range \( \mathbb{I}, 2, 3, \ldots \) which is disjoint from zero. (Can prove this via induction.)

2. \( S \) is one-to-one:
   
   If \( S(n) = S(m) \)
   
   then \( n+1 = m+1 \)
   
   then \( n = m \). \( \square \)

3. The wffs are freely generated from \( \{ A_1, A_2, \ldots \} \)
   
   by \( E =, E v, E A, E \rightarrow, E \leftrightarrow \)
   
   (This is the Unique Readibility Theorem.)

   **Proof**
   
   One-to-one
   
   If \( (\alpha \land \beta) = (\gamma \land \delta) \)
   
   then \( \alpha \land \beta = \gamma \land \delta \) as expressions.

   Either
   
   - \( \alpha \) is a proper initial segment of \( \gamma \)
   
   - \( \gamma \) is a proper initial segment of \( \alpha \)
   
   or \( \alpha = \gamma \)
Hence $\alpha = \gamma$. Similarly $\beta = \delta$.
Therefore $E\alpha$ is one-to-one.

Same for $E\gamma$, $E\nu$, $E\delta$, $E\varepsilon$.

(6) Ranges are disjoint.
Assume $(\alpha \land \beta) = (\gamma \rightarrow \delta)$
By the same reasoning $\alpha = \gamma$
Then $\land \beta) = \rightarrow \delta$)
But $\land \neq \rightarrow$.
Hence $(\alpha \land \beta) \neq (\gamma \rightarrow \delta)$.

Similarly the ranges of $E\gamma$, $E\nu$, $E\delta$, $E\varepsilon$
and the sentence symbols are all disjoint.

$\square$
The unique readability theorem and the recursion theorem show that every truth assignment \( \varphi \) is uniquely extendable to \( \overline{\varphi} \).