A list of important tautologies

1. \( \land, \lor, \iff \) are associative and commutative.
   For example,
   \[
   ((A \land (B \land C))) \iff ((A \land B) \land C))
   \]
   \[
   ((A \land B) \iff (B \land A))
   \]

2. Distributive Laws
   \[
   (A \land (B \lor C)) \iff ((A \land B) \lor (A \land C))
   \]
   \[
   (A \lor (B \land C)) \iff ((A \lor B) \land (A \lor B))
   \]

3. Negation
   \[
   ((\neg \neg A)) \iff A)
   \]
   \[
   ((\neg (A \rightarrow B)) \iff (A \land (\neg B)))
   \]
   \[
   ((\neg (A \iff B)) \iff ((A \land (\neg B)) \lor ((\neg A) \land B)))
   \]

4. De Morgan's laws
   \[
   ((\neg (A \land B)) \iff (\neg A) \lor (\neg B))
   \]
   \[
   ((\neg (A \lor B)) \iff (\neg A) \land (\neg B))
   \]

5. Excluded middle
   \[ (A \lor \neg A) \]
   Contradiction \[ (\neg (A \land (\neg A))) \]
   Contraposition \[ ((A \rightarrow B) \iff (\neg B) \rightarrow (\neg A)) \]
   Exportation \[ ((A \land B) \rightarrow C) \iff (A \rightarrow (B \rightarrow C)). \]
Some basic properties of $t$ =

**Theorem** 1. If $\Sigma \models \alpha$ and $\Sigma \cup \delta \models \beta$, then $\Sigma' \models \beta$.

**Proof** Assume $\Sigma \models \alpha$ and $\Sigma \cup \delta \models \beta$. We want to show $\Sigma \models \beta$.

This, by definition, is the same as showing for all $\nu: \delta \to \{T, F\}$, if for all $\gamma \in \Sigma'$, $\nu(\gamma) = T$, then $\nu(\beta) = T$, where $\delta$ is exactly the set of sentence symbols in $\beta$ and $\Sigma'$.

The problem is that $\delta$ may not contain the sentence symbols from $\alpha$.

**Lemma.** (Exercise #6 on p. 27)

Let $\delta$ be a set of sentence symbols which contain all those in $\Sigma$ and $\beta$ (and possibly more). Then $\Sigma \models \beta$ iff for all $\nu: \delta \to \{T, F\}$, if for all $\gamma \in \Sigma$, $\nu(\gamma) = T$, then $\nu(\beta) = T$.
(proof continued...)

Let $\mathcal{S}'$ be the sentence symbols in
$E$, $\alpha$, and $\beta$.
Let $\nu: \mathcal{S'} \to \{T, F\}$ be an arbitrary truth
assignment.

By the lemma, it is enough

Assume for all $\gamma \in E$ that $\nu(\gamma) = T$.
Then, since $E \vdash \alpha$, we have $\nu(\alpha) = T$
(using the Lemma since $\mathcal{S}'$ may have more
sentence symbols than $E$ and $\alpha$).
Then for all $\gamma \in E \cup \{\alpha\}$, $\nu(\gamma) = T$
since either $\gamma \in E$ or $\gamma = \alpha$.
Since $E \cup \{\alpha\} \vdash \beta$, then $\nu(\beta)$, as desired.
Therefore, $E \vdash \beta$. \hfill $\square$

Similarly, we can show the following:

**Corollary** If $E \vdash \alpha$ and $\alpha \vdash \beta$, then $E \vdash \beta$.

**Proof** Assume $E \vdash \alpha$ and $\alpha \vdash \beta$.
Then $\alpha \vdash \beta$.
By the previous theorem (with $E = \emptyset$)
we have $\vdash \beta$. \hfill $\square$
Induction and Recursion

In mathematics and computer science, there is a special type of way to define a set of objects.

1. Take a large set of objects $U$.
2. Take a small subset of objects $B \subseteq U$.
3. Let $C$ be the smallest subset of $U$ containing $B$ that is closed under all the operations in some class $F$.

(Such a definition is called a recursive definition or sometimes an inductive definition.)

Examples

1. Natural numbers
   \[ U = \mathbb{N} \]
   \[ B = \{0\} \]
   \[ F = \{S\} \quad \text{where} \quad S(n) = n+1 \]

2. Integers
   \[ U = \mathbb{Z} \]
   \[ B = \{0\} \]
   \[ F = \{S, P\} \quad \text{where} \quad S(n) = n+1 \quad \text{and} \quad P(n) = n-1 \]

   Notice $0$, $S(P(0))$, and $P(S(0))$ are equal.

3. Wffs
   \[ U = \text{set of expressions} \]
   \[ B = \text{set of sentence symbols} \]
   \[ F = \{\mathcal{E}, \mathcal{E}_v, \mathcal{E}_w, \mathcal{E}_r\} \]
For simplicity, let us assume \( \mathcal{F} = \{ f, g \} \) 
where \( f: U \times U \to U \) and \( g: U \to U \)

Two ways to define \( C \):

**From the top down**

Say \( S \subseteq U \) is closed under \( f \) and \( g \) iff

whenever \( x, y \in S \) then \( f(x, y) \in S \) and \( g(x) \in S \)

Say \( S \subseteq U \) is inductive iff

1. \( B \subseteq S \) and
2. \( S \) is closed under \( f \) and \( g \).

Let \( C^* = \bigcap \{ S \subseteq U \mid S \text{ is inductive} \} \)

_Thm \( C^* \) is inductive_

Proof: We need to show both conditions.

1. **\( B \subseteq C^* \)**
   
   Assume \( x \in B \), then for all inductive \( S \), \( x \in S \) since \( B \subseteq S \).
   
   Therefore \( x \in C^* \) by the definition of \( \cap \).
   
   Hence \( B \subseteq C^* \).

2. **\( C^* \) is closed under \( f \) and \( g \)**

   Assume \( x, y \in C^* \).

   Let \( S \) be an arbitrary inductive set.

   Since \( C^* \subseteq S \), \( x, y \in S \).

   Then \( f(x, y), g(x) \in S \).

   Therefore \( f(x, y), g(x) \in C^* \) since \( S \) was arbitrary.

   Hence \( C^* \) is closed under \( f \) and \( g \). \( \square \)