I was not until my 4th semester of graduate school that I learned how to study for a proof-based mathematics course (say a 400 level course or higher at Penn State). Mind you, I could get good grades—often A’s—through a combination of reading just enough of the book or lecture notes to do the homework (without understanding the material as a whole) and by cramming for exams. When I got to graduate school I realized I did not know much of the material that I supposedly learned. This was a frightening realization! Through trial and error I came up with a workflow that allowed me to better learn and retain the material. While I may have more difficulty in following a math lecture than the average individual\(^1\), I believe there is a great need among the average mathematics student, and even among talented undergraduate and graduate students, for advice on how to get the most out of your mathematics course. In this note, I will describe the method that worked for me.

1 What I was doing wrong

Here is a series of misconceptions I once held. From talking with others, I realized I am not alone in having once held these beliefs.

1. I cannot master the material if I have a poor instructor.

2. I cannot master the material if I have little or no homework (or no homework solutions).

3. I cannot master the material if the lectures are difficult to follow.

4. If the lectures are difficult to follow, then either the professor is a poor instructor, or worse, I am bad at math.

5. If I am behind in lecture, I cannot get caught up.

6. It is enough to be able to do the homework to learn the material.

7. Proofs are not important. I only need to know the definitions and theorems.

\(^{1}\)I may have ADHD, although I have never sought a diagnosis.
Since the lecture is mostly proofs, most of the lecture is not important either.

If I got an A in the class, then I learned the material.

2 The method that worked for me

The main idea is to wrestle with the notes after class. But this must be done correctly. I will also mention an alternative, which is to wrestle with the book or lecture notes before class.

2.1 The method: Wrestle with the notes after class

Step 1: Copy down the lectures. Virtually every mathematics instructor writes a set of lecture notes on the chalkboard. Write down everything, even if it makes it more difficult to follow the lecture. What I am about to say next is going to sound heretical.

It is okay if you get lost in lecture. It is even okay if you are lost for the majority of lectures.

Why? This is because mathematics takes time to absorb. Absorbing mathematics is like reading Middle English poetry, for example “The lyf so short, the craft so long to lerne.” I suspect it took you two readings to even parse that sentence, and it took another reading to understand its meaning.

Step 2: Prior to the next lecture, read—no, wrestle with—the lecture notes. In my experience this will take about an hour, the same amount of time the lecture took. It will go faster if you understood the lecture, slower if you did not. You will be surprised how beneficial this exercise is. When wrestling though the notes, you will gain a better understanding of the definitions, theorems, and proofs. The flow of the lecture will make sense. That off-hand comment that the professor made, will be more clear. That mystery word on the board (that looks like “ksdjjfsj) will become clear from context, as will typos and small errors on the part of the professor. When you are done, you will really feel as if you understand the material. This, of course, assumes you are reviewing the material in the correct manner. See the next section for how to wrestle with mathematics.

Step 3: Make this routine a habit. Commit to reviewing every lecture before the next one. You will always find excuses not to study. For this method to work, you will need to set aside a regular time to do your class review. I would often did my reviews either right after class or I would set aside an hour during the next day.

Step 4: If you get behind (and you will at some point), cram to get caught up. By “cram” here, I mean do the same thing that you would do before an exam when you do not have enough time to properly study. If you do not have an hour to look over your notes, then find ten minutes to read the most pressing stuff, usually the definitions and theorems. If you are a week behind, then possibly start by reviewing the most recent lectures, looking back at the older material as time allows.

2.2 How to properly “wrestle with mathematics”

Properly reviewing your notes can be at least as beneficial as homework, and usually more so. However, it is not enough to just glance at the material. Reading mathematics is not passive. You cannot lie down in bed with your lecture notes and read them as if they were a paperback novel (although we have all tried!). Reading mathematics is at best a conversation with the text, and at worse, a heated argument. You must be alert. You must question the text, and at times even disagree with it.

Be prepared. Set aside an hour (or an hour and a half just in case). Sit down at a table or desk in a location that is not distracting. Bring along a pencil and lots of scratch paper. You cannot wrestle with math without the ability to write notes, draw pictures, and make annotations!

Definitions. When you get to a definition, memorize it. Ask yourself these questions: Why is this definition important? (Often the answer may not be clear until a few lectures later, but ask yourself anyway.) Are there any obvious examples? Is this similar to other definitions you know? What are some basic properties of this definition?

A big advantage of reading notes on your own time is that you can look up words or notation from earlier in the course (or from previous courses) that you have forgotten. The same goes for notation that you have never seen before! (The internet is a wonderful resource.) As a corollary to this, make sure you have all the course notes from earlier in the semester easily accessible to you. I personally keep all my notes for a given course together in single spiral-bound notebook.

Theorems (and Lemmas, Propositions, etc.). When you get to a theorem, memorize it. Ask yourself these questions: What is important about this theorem? Is it a lemma used to prove something more important? Is it a basic property about a new definition? Is it an important tool that will be used later in the course? Is it an interesting fact in itself? Why are the conditions of the theorem there? Are there clear counterexamples if the conditions are weakened? If the theorem is an implication, is the converse true as well?
Proofs. Usually the most difficult part about review notes is following the proofs. When you get to a proof, try to prove it yourself, especially if you vaguely remember it from lecture or if you think the proof is routine. To do this, cover up the proof. Then write the proof (or a proof sketch, or even a proof picture) on scrap paper. If you get stuck, peek at your notes to get the main idea, and then cover them up and try to finish the proof. This way, you will make sure your really know the proof.

If the proof is too difficult, then just read through the proof. If you get to a step you do not understand, try to work it out for yourself. If you are still stuck, then make a note of it and return to it later, possibly asking a classmate or the professor for help. After you get through a long proof, ask yourself what the main idea of the proof was. It is easy to get lost in the details and “miss the forest for the trees”. This is especially true in computational proofs, or in long proofs that take two or more lectures to prove.)

Annotating and correcting your notes. If you come across a typo, it will probably be clear from context what the professor meant. Correct those. Also, take note of anything confusing in your notes. When you figure it out, annotate your notes. (I write annotations in the margin with bubbles around them to signify that they were added later). This will help you when you look over your notes again. Also add annotations if you have an insightful observations.

Let your curiosity run wild. Last, feel free to let your curiosity run wild. Feel free to investigate new definitions by thinking of examples, or deriving basic properties. Also think of connections between what you are learning in this class and what you learned in other classes. (Many of these connections go unmentioned because the professor does not want to confuse the students who have never seen that related material.)

2.3 An alternative method: Wrestle with the textbook or the lecture notes before class

This alternate option works if the class closely follows either a textbook or a set of prewritten lecture notes. It is also useful if the lectures are particularly difficult to follow and the professor does not write much on the board. The key here is that you want to read ahead of the lecture so that you can compare the lecture to what you read.

Step 1: Figure out what material will be covered in the next lecture. This step is difficult. One option is to ask the professor for a rough estimate of what will be covered. Note, some professors do not know what they will cover until a few hours before class. Another option is to make an educated guess based on where you are in the book already. Yet another option is to attempt to be so far ahead that you will, for sure, have read whatever will be covered in lecture.
Step 2: Wrestle with the appropriate section of textbook before the next lecture. Follow the same advice as for wrestling with course notes. It is even possible that books—and especially course notes—have typos, so take nothing for granted. (To help with the typos, find an errata list—a list of typos and errors—for the book. Google “<author name> errata”. The errata list is usually maintained on either the author’s website or on the publisher’s website.)

One difference between reading textbooks vs. class notes is that with textbooks you will not always know how fast reading should go. With course notes you know about how long it takes to review one lecture’s worth of notes. With a book it may take an hour to read three pages or an hour to read just half a page. And it will vary from page to page in the textbook. Do not be dismayed. If it takes you an hour to get through half a page in the textbook, it will probably take the teacher a whole lecture to get through the same material.

Step 3: Compare the textbook to the lecture. For this step, it is important that you are reading ahead of lecture! When you get to the lecture, make sure the teacher is following the book. Do they use different notation? Do they provide different proofs? Do they add new material or skip material from the book? If they present material that is not in the book, then make sure you take good lecture notes that day and review them after class!

Lecture is also a good time to ask about things you were confused about in the reading. If it was still confusing after the professor mentions it, then ask. (You may impress your professor with how fast you are able to pick up the material.)

Step 4: Review anything you did not understand in lecture. This is really important. Just because you read ahead, does not mean you are immune to being confused in lecture.

Step 5: Keep up the habit, and catch up (through cramming) when you get behind. This is the same as reading lecture notes. Make this a habit, with a regular time for reading. Also, if you get behind, cram to get caught up. For example, ten minutes of reading ahead is better than none if that is all the time you have available before class.

3 Homework, exams, reading papers

3.1 Homework

While homework is not the end-all-be-all of mathematical learning, it is still a helpful tool for learning mathematics.

Start your homework as early as possible, preferably the day it is assigned. Mathematical proofs are not like other types of homework. They are similar to riddles and puzzles. There is not always a straightforward method
to solve them. They require an insight or trick to solve. (This is in stark contrast to calculus problems!) The typical routine to come up with a mathematics proof is as follows.

Read the problem. Try to prove it. Hit a wall. Take a break. Repeat.

Then—often during one of the breaks—you will have an eureka moment. However, for this method to work, you will need to start early.

**Know the subject matter before you start the homework.** A misguided, although all-too-common, technique to solving homework problems is to read just enough of the book to do each problem. More specifically, one takes out the homework assignment, reads the first problem, and then looks in the book or the notes for exactly the definition, theorem or remark that lets one solve that problem. There are a number of issues with this method: It is a terribly inefficient way to learn the material. It will leave many gaps in your understanding of the material. (It is difficult for the professor to come up with a homework assignment for every concept taught in class.) You may end up doing the problem the hard way. For example, you may solve a problem by working through the definitions, when the “correct method” is to use a later theorem which greatly simplifies the problem.

**Do not be a leech to your classmates.** Try the problems on your own before working with others. When you work with others, come prepared to provide helpful insights and suggestions. Also make sure you understand any solution you write down. In other words, copying is a horrible way to learn the material (and it is cheating!).

**Be your own worst critic.** One of the most important mathematical talents is to know a bad proof when you see one. Practice this skill on yourself. Proof-read your homework and make sure your proofs are valid. If you are unsure about a step, then try to work it out further until you have justified it to yourself.³ Ask the professor if you are still unsure. A small error—such as reversing an inequality, or using “A → B” when the theorem actually says “B → A”—can make an argument go from “all correct” to “all wrong”. You are not doing anyone a favor by trying to “pull a fast one” over yourself or the grader. Be brutally honest; if you have a step in your homework that you cannot justify, then make a note in the margin asking if this is correct. (When I was a student, I even did the same thing even on exams!) Why would you do this? Because you care more about learning the material than about a point off your homework grade.

³For example, you may want to use the rule $|y - x| \geq ||y| - |x||$, but you are unsure about the direction of the inequality. You can test it on a basic example like $x = -1$ and $y = 1$ to see if you are correct. If you are still unsure, then look it up, or better yet, rederive it using the triangle equality in the following form $|x + (y - x)| \leq |x| + |y - x|$. 
Be professional and neat. Turn in a final draft, not a first draft. Use legible handwriting. Space out the problems (possibly a new page for each problem), and put them in the correct order. Write down the problem statement along with the problem. Be clear and precise in your notation and your reasoning. Cite the results and references that you use. Use complete sentences (or complete mathematical sentences), including periods. Do not abuse notation or use excessive shorthand. Staple your homework. Do not include scratch work.

Scan or photocopy your homework before turning it in. Professors are notorious for losing homework, or for taking forever to grade it.

Treat the homework as an extension of the lecture. Homework problems are assigned for a variety of reasons. Often they are important facts or examples that could have been proved in lecture, but were given as homework instead. Do not just solve the problem and move on, ask yourself why that problem was important. Was it a useful fact which can be used later? Was it an interesting example of a concept learned in class? Did it allow you to practice a technique from class or understand a definition better? Remember, homework problems are theorems as well, and sometimes they are important theorems.

3.2 Exams
Exams (both in-class or take-home) are given so that you will be forced to review the material. Use the “dread” of the final exam as a motivation to keep up the habit of studying and reviewing the material throughout the semester. As the exam draws near, go over the material again in a systematic manner. Look at the big picture. Make a list of topics covered. Review the topics that you do not remember as well. Review the theorems, definitions, and proofs from lecture. Also look over the homework problems.

It is okay to study “too much”. It is a wonderful feeling to leave an exam knowing that you have aced it (and knowing that you learned the material well because of all your studying).

3.3 Reading papers
Some mathematics classes assign papers to read (or chapters from a book). As with homework, start early. This is often more time consuming than you may think. When reading a paper, follow the same advice as when reading a textbook or when reviewing your lecture notes. Wrestle with the text!

However, articles can differ from books in a few key ways. Articles are usually more dense. It will take longer to read a page in an article that a page in a textbook. Unlike a book, an article may not be self contained. You may need to go elsewhere for background. In some cases the author may not have provided a reference for the background material that he/she felt was “too basic”. The notation may also differ from what you are used to. Do not let all these things
dismay you. Wrestle through the paper a little at a time. Eventually, you will understand what it means.

Last, do not be shy about asking for help. Your professor will be more than glad to assist those who come early with questions.

4 Do not overdo it!

It is possible to go too far and to burn yourself out. Here are some examples of where I have went too far in the past. I know other people have done these activities and found them to be useful, so take this advice as you will. Just remember, it is better to do a medium amount of work consistently than to do a large amount of work inconsistently. The key is to distinguish “busy work” from actual studying.

1. I found I could not keep up with both reading the book before class and reviewing my notes after class. It took too much time to do both. I often chose to just review the class notes after class since they followed the lecture more closely than the book.

2. When reviewing a lecture, I could overthink every definition and theorem and proof. If I was not careful, I could spend three hours reviewing a single lecture. I found the key was to focus on the main material and try to get done in 1–1.5 hours.

3. I have experimented with rewriting my lecture notes, even using \texttt{L\LaTeX} to type them up. It was quite time consuming, and worse I would sometimes get into a habit of rewriting the notes without ever thinking about what they said!

4. For some classes, I would type my homework in \texttt{L\LaTeX}. However, I realized it was taking time from more important activities, such as reviewing the material. Also, I found I was more likely to have a mistake in a solution that I typed up. (That being said, if you plan to go to graduate school in mathematics or computer science, you will eventually need to learn \texttt{L\LaTeX}. Typing up your homework is one of the best ways to learn \texttt{L\LaTeX}.)

5. I used to highlight my textbooks: definitions in yellow, theorems in green, etc. However, math books are already “highlighted”: definitions are in italics, theorems are set apart and easy to find. I realized my time was much better spent wrestling with the text than by making it pretty with colors.