Randomness for capacities
with applications to random closed sets

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Computability, Complexity, and Randomness
June 2014

Slides available at
www.personal.psu.edu/jmr71/

(Updated on June 17, 2014.)
Introduction and Motivation
The main idea

- All measures are additive:
  \[ \mu(A \cup B) = \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint}) \]

- However there are also nonadditive “measures”, in particular
  - subadditive “measures” (a.k.a. capacities)
    \[ \mu(A \cup B) \leq \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint}) \]
  - superadditive “measures”
    \[ \mu(A \cup B) \geq \mu(A) + \mu(B) \quad (A \text{ and } B \text{ disjoint}) \]

- Martin-Löf randomness can be extended to nonadditive “measures”.

- This provides a unified framework for many notions in randomness.
Examples of randomness for capacities (subadditive)

- strong $s$-randomness (resp. strong $f$-randomness) … (Reimann and many others)
  \[ \text{randomness on $s$-dimensional (resp. $f$-weighted) Hausdorff capacity} \]
- $s$-energy randomness \[ (\text{Diamondstone/Kjos-Hanssen}) \]
  \[ \text{randomness on $s$-dimensional Riesz capacity} \]
- MLR for a class of measures \[ (\text{Bienvenu/Gács/Hoyrup/Rojas/Shen}) \]
  \[ \text{randomness on the corresponding upper envelope capacity} \]
- members of a MLR closed set
  - MLR closed sets \[ (\text{Barmpalias/Brodhead/Cenzer/Dashti/Weber}) \]
  - zeros of MLR Brownian motion \[ (\text{Kjos-Hanssen/Nerode and A/B/S}) \]
  - image of MLR $n$-dim. Brownian motion \[ (\text{Allen/Bienvenu/Slaman}) \]
  - double points of MLR planar BM \[ (\text{Allen/Bienvenu/Slaman}) \]
  \[ \text{randomness on the corresponding intersection capacity} \]
- (Unfinished work) Lebesgue points of all computable Sobolev $W^{n,p}$ functions
  \[ \text{(some sort of) Schnorr randomness on $n,p$-Bessel capacity} \]
More examples

- Randomness for superadditive measures loosely corresponds to
  - randomness for semimeasures (studied by Levin and Bienvenu/Hölzl/Porter/Shafer)

- Randomness for capacities can be used to characterize
  - effective dimension (studied by Lutz and everyone else)
An application: members of random closed sets

Random closed sets of Barmpalias/Brodhead/Cenzer/Dashti/Weber

- Construct a \{0, 1\}-tree with no dead-ends:
  - Branch only left with probability 1/3.
  - Branch only right with probability 1/3.
  - Branch both directions with probability 1/3.
- A MLR tree is one constructed with a MLR in \( 3^\mathbb{N} \).
- A **BBCDW MLR closed set** is the set of paths through a MLR tree.
- What are the elements of a BBCDW MLR closed set?

**Theorem (Diamondstone and Kjos-Hanssen)**

- If \( z \) is MLR on some probability measure \( \mu \) satisfying the condition
  \[
  \iint |x - y|^{-\log_2(3/2)} \, d\mu(x) d\mu(y) < \infty,
  \]
- then \( z \) is a member of some BBCDW MLR closed set.

- D/K-H conjectured the converse holds. **I will show they were correct!**
Definitions and Basic Results
**Nonadditive measures**

**Definition**

A **regular nonadditive measure** on $2^\mathbb{N}$ or $\mathbb{R}^n$ is a set function $C$ such that

- $C$ is defined on all open and closed sets,
- $C$ is finite on compact sets,
- $C(\emptyset) = 0$,
- $C$ is monotone: $C(A) \leq C(B)$ for $A \subseteq B$,
- $C$ continuous from below on open sets: $C(U_n) \uparrow C(U)$ if $U_n \uparrow U$,
- $C$ continuous from above on compact sets: $C(K_n) \downarrow C(K)$ if $K_n \downarrow K$,
- $C$ is outer regular: $C(A) = \inf\{C(U) : U$ open and $A \subseteq U\}$,
- $C$ is inner regular: $C(A) = \sup\{C(K) : K$ compact and $K \subseteq A\}$.

**Definition**

If $C$ is subadditive ($\mu(A \cup B) \geq \mu(A) + \mu(B)$) then call $C$ a **capacity**.
## Computable nonadditive measures

### Definition

Say that a regular nonadditive measure is **computable** if

- $C$ is uniformly lower-semicomputable on $\Sigma_1^0$ sets, and
- $C$ is uniformly upper-semicomputable on $\Pi_1^0$ sets.

Or equivalently on $2^{\mathbb{N}}$,

- $C$ is uniformly computable on clopen sets (not just cylinder sets).

### Definition

A **Martin-Löf test** $(U_n)$ is a uniform sequence of $\Sigma_1^0$ sets such that $C(U_n) \leq 2^{-n}$.

A point $x$ is **Martin-Löf random** on $C$ if $x \notin \bigcap_n U_n$ for all ML tests.

### Remark

If $C$ is countably subadditive, $C(\bigcup_n A_n) \leq \sum C(A_n)$,

- There is a universal test (same proof!).
- Can also use Solovay tests, integrable tests, etc.
An important lemma

For two capacities, write

\[ C_1 =^\times C_2 \]

if they are equal up to a multiplicative constant, i.e. there are two constants 0 < a < b such that for all A,

\[ a C_1(A) \leq C_2(A) \leq b C_1(A). \]

Lemma

For two computable capacities, if \( C_1 =^\times C_2 \) then \( C_1 \) and \( C_2 \) have the same ML randoms.
Randomness for non-computable measures

- It is well known that MLR can be extended to noncomputable probability measures (Levin, Reimann/Slaman, Day/Miller, B/G/H/R/S)

**Definition**

A point $x \in X$ is **Martin-Löf random on** $\mu$ if $x$ is not covered by any ML tests “computable from $\mu$”.

- Let $\mathcal{M}^+(X)$ denote the space of all finite Borel measures on $X$.

**Lemma**

If $x$ is random on $\mu \in \mathcal{M}^+(X)$, then $x$ is random on the prob. measure $\mu/\mu(X)$.

- Randomness can also be developed for non-computable capacities, but this is not needed in this talk.
Effective compactness

On any computable metric space $X$ (e.g. $2^\mathbb{N}$, $\mathbb{R}^n$, $\mathcal{M}^+(2^\mathbb{N})$, $\mathcal{M}^+(\mathbb{R}^n)$) let $\mathcal{K}(X)$ denote the space of nonempty compact sets.

Definition

A nonempty compact set $K$ is said to be $^1$computable in $\mathcal{K}(X)$ iff any of the following equivalent conditions hold:

1. $K$ is computable in the Hausdorff metric on $\mathcal{K}(X)$.
2. The distance function $x \mapsto \text{dist}(x, K)$ is computable.
3. $K$ is the image of a total computable map $2^\mathbb{N} \to X$.
4. $K$ is $\Pi_1^0$ and contains a computable dense subsequence.
5. (On $2^\mathbb{N}$) $K$ is the set of paths of a computable tree with no dead branches.
6. (On $\mathbb{R}$) Both $\{a, b \in \mathbb{Q} : K \cap (a, b) \neq \emptyset\}$ and $\{a, b \in \mathbb{Q} : K \cap [a, b] = \emptyset\}$ are $\Sigma_1^0$.
7. The map $f \mapsto \max(K, f)$ is uniformly computable for continuous $f$.

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$^1$Others use the terms “located set”, “computable set”, or “decidable set”. The term “effectively compact set” is usually reserved for a weaker notion.
Some Examples of Capacities and Characterizations of their Random Points
Let $\mathcal{C}$ be a compact subset of $\mathcal{M}^+(X)$ (finite Borel measures on $X$).

The **upper envelope capacity** is defined as

$$\text{Cap}_\mathcal{C}(A) = \sup_{\mu \in \mathcal{C}} \mu(A).$$

**Theorem**

*If $\mathcal{C}$ computable in $\mathcal{K}(\mathcal{M}^+(X))$, then $\text{Cap}_\mathcal{C}$ is a computable capacity.*

**Theorem (Basically Bienvenu-Hoyrup-Gács-Rojas-Shen)**

*If $\mathcal{C}$ computable in $\mathcal{K}(\mathcal{M}^+(X))$, the following are equivalent.*

- $x$ is random on the upper envelope $\text{Cap}_\mathcal{C}$.
- $x$ is random on some measure $\mu \in \mathcal{C}$.
Riesz capacity / Energy randomness

- \( s \)-energy of a measure \( \mu \in \mathcal{M}^+ (\mathbb{R}^n) \)

\[
\text{Energy}_s(\mu) := \begin{cases} 
\iint |x-y|^{-s} d\mu(x)d\mu(y) & 0 < s \leq n \\
\iint \log \frac{1}{|x-y|} d\mu(x)d\mu(y) & s = 0
\end{cases}
\]

- \( s \)-Riesz capacity (version 1)

\[
C_s(A) := \sup \{ \mu(A) | \mu \in \mathcal{M}^+ (\mathbb{R}^n), \text{Energy}_s(\mu) \leq 1 \}
\]

**Theorem**

For computable \( s \),

- \( \{ \mu | \mu \in \mathcal{M}^+ (\mathbb{R}^n), \text{Energy}_s(\mu) \leq 1 \} \) is a computable set in \( \mathcal{K} (\mathcal{M}^+ (\mathbb{R}^n)) \).
- \( C_s \) is a computable (upper envelope) capacity

The following are equivalent:

- \( x \) is random on the Riesz capacity \( C_s \).
- \( x \) is random on some measure \( \mu \in \mathcal{M}^+ (\mathbb{R}^n) \) such that \( \text{Energy}_s(\mu) \leq 1 \).
- \( x \) is random on some probability measure \( \mu \) such that \( \text{Energy}_s(\mu) < \infty \)

\( \Leftrightarrow \): \( s \)-energy random.
Some Examples of Capacities and Characterizations of their Random Points

Riesz capacity / Energy randomness

- **s-Riesz capacity (version 1)**
  \[ C_s(A) := \sup \{ \mu(A) \mid \mu \in \mathcal{M}^+(\mathbb{R}^n), \text{Energy}_s(\mu) \leq 1 \}. \]

- **s-Riesz capacity (version 2)**
  \[ \text{Cap}_s(A) := \sup \left\{ \frac{1}{\text{Energy}_s(\mu)} \mid \mu \in \mathcal{M}_1(A) \right\}. \]

**Theorem**
\[ \text{Cap}_s = (C_s)^2. \]

**Corollary**
The following are equivalent
- \( x \) is random on the Riesz capacity \( \text{Cap}_s \).
- \( x \) is random on the Riesz capacity \( C_s \).
- \( x \) is \( s \)-energy random.
Some Examples of Capacities and Characterizations of their Random Points

Random compact sets / Members of MLR compact sets

- For a probabilist, a **random set** is a process which randomly generates a set, i.e. a set-valued random variable $Z$.

- Each random compact set corresponds to a probability measure $P_Z$ on the space of compact sets $\mathcal{K}(2^\mathbb{N})$.

- A compact set $K$ is a **MLR compact set** if it is MLR on $P_Z$.

- The **intersection capacity** is defined as follows:

  \[ T_Z(A) = P\{Z \cap A \neq \emptyset\} = P_Z\left\{ K \in \mathcal{K}(2^\mathbb{N}) : K \cap A \neq \emptyset \right\}. \]

  $T_Z(A)$ is the probability that $A$ intersects some random compact set.

**Theorem (R.)**

*For computable measure $P_Z$ on $\mathcal{K}(2^\mathbb{N})$, the following are equivalent.*

- $x$ is a random for the intersection capacity $T_Z$.
- $x$ is a member of some $P_Z$-MLR compact set.
Applications of Potential Theory to Random Compact Sets
Proof of Diamondstone and Kjos-Hanssen’s conjecture

Conjecture (Diamondstone/Kjos-Hanssen)

The following are equivalent.

- $x$ is a member of some BBCDW MLR compact set.
- $x$ is $\log_2(3/2)$-energy random.

- Let $T_{\text{BBCDW}}$ be the intersection capacity of the BBCDW random set.
- Recall $\text{Cap}_{\log_2(3/2)}$ denotes the $\log_2(3/2)$-Riesz capacity.

Equivalent Conjecture

$T_{\text{BBCDW}}$ and $\text{Cap}_{\log_2(3/2)}$ have the same randoms.

The conjecture follows from...

Theorem (Lyons)

$T_{\text{BBCDW}} \times \text{Cap}_{\log_2(3/2)}$. 
The zeros of Brownian motion $B$ form a random closed set.

$$Z_B = \{ t \in [a,b] : B(t) = 0 \}$$

**Question (Allen/Bienvenu/Slaman)**

Which reals are zeros of some MLR Brownian motion?
Zeros of a MLR Brownian motion

Let $T_{Z_B}$ be the intersection capacity of the random closed set $Z_B$ of zeros.

$$T_{Z_B}(A) := \mathbb{P}\{Z_B \cap A \neq \emptyset\}$$

**Theorem (Kakutani)**

$T_{Z_B} = \times \text{Cap}_{1/2}$ when restricted to an interval $[a, b]$ ($0 < a < b$).

**Corollary**

$T_{Z_B}$ and $\text{Cap}_{1/2}$ have the same randoms (in $[a, b]$)

**Theorem (R.—Allen/Bienvenu/Slaman were very close!)**

The following are equivalent.

- $x$ is a zero time of some MLR Brownian motion.
- $x$ is $1/2$-energy random.
Image of an $n$-dimensional Brownian motion

The image of an $n$-dimensional Brownian motion forms a random closed set.

$$B([a,b]) = \{B(t) : a \leq t \leq b\}$$

**Question (Allen/Bienvenu/Slaman)**

Which points are in the image of some $n$-dimensional MLR Brownian motion?

**Figure**: Image of a 3D Brownian motion (http://en.wikipedia.org/wiki/Wiener_process)
Image of a MLR $n$-dimension Brownian motion

Let be the intersection capacity of $B([a,b])$.

$$T_{B([a,b])}(A) := \mathbb{P}\{B([a,b]) \cap A \neq \emptyset\}$$

**Theorem (Kakutani)**

$$T_{B([a,b])} \equiv \times \text{Cap}_{n-2} \text{ when } n \geq 2.$$  

**Corollary**

$T_{B([a,b])}$ and $\text{Cap}_{n-2}$ have the same randoms (for $n \geq 2$).

**Theorem (R.)**

The following are equivalent for $x \in \mathbb{R}^n$ ($x \neq 0$) and $n \geq 2$.

- $x$ is in the image of some $n$-dimensional MLR Brownian motion.
- $x$ is $(n-2)$-energy random.
Double points of planar Brownian motion

- The double points of a planar Brownian motion are the points where it intersects itself.

**Question (Allen/Bienvenu/Slaman)**
Which points are the double points of some MLR planar Brownian motion?

- Closely related, consider the intersection points of two independent planar Brownian motions which start at the origin.
- This forms a random closed set.

**Figure**: Intersection (in blue) of two planar Brownian motions starting at the origin.
Double points of MLR planar BM

- Let $T_{B_1 \cap B_2}$ be the intersection capacity of random closed set $B_1([a,b]) \cap B_2([a,b])$.

$$T_{B_1 \cap B_2}(A) := \Pr\{B_1([a,b]) \cap B_2([a,b]) \cap A \neq \emptyset\}$$

- Define $\text{Cap}_{\log^2}$ and log$^2$-energy random using the energy

$$\iint (\log|x-y|)^2 \text{d}\mu(x)\text{d}\mu(y).$$

Theorem (Fitzsimmons/Salisbury, also Permantal/Peres)

$T_{B_1 \cap B_2} \propto \text{Cap}_{\log^2}$. 

Theorem (R.)

The following are equivalent for $x \in \mathbb{R}^2$ ($x \neq 0$).

- $x$ is a double point of some MLR planar Brownian motion.
- $x$ is in the intersection of some pair of independent MLR planar BM.
- $x$ is log$^2$-energy random.
Capacities and Effective Hausdorff Dimension
Hausdorff capacity vs Hausdorff measure

- **s-Hausdorff capacity** (a.k.a. **s-Hausdorff content**) of radius \( r \in (0, \infty] \).

\[
\mathcal{H}_r^s(A) := \inf \sum_i 2^{-s|\sigma_i|}
\]

where the infimum is over covers \( \bigcup_i [\sigma_i] \supseteq A \) such that \( 2^{s|\sigma_i|} \leq r \).

- **s-dimensional Hausdorff measure**

\[
\mathcal{H}^s(A) := \lim_{r \to 0} \mathcal{H}_r^s(A)
\]

- Same null sets:

\[
\mathcal{H}^s(A) = 0 \iff \mathcal{H}_r^s(A) = 0.
\]

- \( \mathcal{H}^s \) is an ugly measure (open sets have infinite measure!)

- \( \mathcal{H}_r^s \) is a nice capacity.

- \( \mathcal{H}_\infty^s \) is called **vehement weight** by some in algorithmic randomness.
Frostman’s Lemma / Strong s-randomness

Frostman’s Lemma

The following are equivalent.

- $H^s_\infty(A) > 0$
- $\mu(A) > 0$ for some prob. measure $\mu$ such that $\mu(\sigma) \leq \times 2^{-s|\sigma|}$ for all $\sigma$.

- A point $x \in 2^\mathbb{N}$ is strongly $s$-random if $KM(x \upharpoonright n) \geq sn + O(1)$. ($KM$ is a priori complexity.)
- A point $x \in 2^\mathbb{N}$ is vehemently $s$-random if $x$ is $H^s_\infty$-random.
- A point $x \in 2^\mathbb{N}$ is $s$-capacitable if $x$ is $\mu$-random for some prob. measure $\mu$ s.t. $\mu(\sigma) \leq \times 2^{-s|\sigma|}$ for all $\sigma$.

Effective Frostman’s Lemma (Reimann, Kjos-Hanssen)

The following are equivalent.

- $x$ is strong $s$-random.
- $x$ is $s$-vehemently random ($H^s_\infty$-random).
- $x$ is $s$-capacitable.
Effective dimension (Frostman’s Theorem)

Theorem (Frostman’s Theorem)

\[ \text{dim}(A) = \sup \{ s \mid \mathcal{H}^s_r(A) > 0 \} = \sup \{ s \mid \text{Cap}_s(A) > 0 \} \]

Theorem (Effective Frostman’s Theorem (Reimann, Diamondstone/K-H))

\[ \text{cdim}(x) = \sup \{ s \mid x \text{ is strongly } s\text{-random} \} = \sup \{ s \mid x \text{ is } s\text{-energy random} \} \]

Question (Reimann)

Are strong \( s \)-randomness and \( s \)-energy randomness the same?

Answer

No.
Question

Are strong $s$-randomness and $s$-energy randomness equal?

Theorem (Maz’ya/Khavin, also in Adams/Hedberg book)

There is a(n effective) closed set $E$ such that $\mathcal{H}^s_r(E) > 0$, but $\text{Cap}_s(E) = 0$.

Corollary (R.)

There is an $x$ which is strongly $s$-random, but not $s$-energy random.
## Summary of known results

### Known Results (No arrows reverse)

- \( \text{cdim } x > s \)
- \( \Rightarrow x \text{ is } s\text{-energy random} \iff x \text{ is random on } \text{Cap}_s \)
- \( \Rightarrow x \text{ is strongly } s\text{-random} \iff x \text{ is random on } \mathcal{H}_r^s \)
- \( \Rightarrow x \text{ is weakly } s\text{-random} \) (that is \( K(x \upharpoonright n) \geq sn + O(1) \))
- \( \Rightarrow \text{cdim } x \geq s \).

### Remark

Weak \( s\)-randomness is not associated (at least in any nice way) with a capacity.
Let $T_{BBCDW}$ be the intersection capacity of the BBCDW random sets. Let $H_{\infty}^{\log_2(3/2)}$ be the Hausdorff capacity for dimension $s = \log_2(3/2)$. It turns out that

- $T_{BBCDW}([\sigma]) = H_{\infty}^{\log_2(3/2)}([\sigma]) = (2/3)|\sigma|$. 
- $T_{BBCDW}$-random (log$_2$(3/2)-energy random) $\neq H_{\infty}^{\log_2(3/2)}$-random (strong log$_2$(3/2)-random).

**Remark**

Therefore, the values of a capacity on cylinder sets is not enough to determine its randomness.
Superadditive Measures and Semi-measures
Semi-measures

- Semi-measures:
  \[ \rho(\sigma_0) + \rho(\sigma_1) \leq \rho(\sigma). \]

- Levin and Bienvenu/Hölzl/Porter/Shafer have looked at randomness for computable and left c.e. semimeasures.
- It is messy in the left c.e. case!

- Can extend \( \rho \) to the smallest superadditive measure \( C \geq \rho \).
- **Blind (Hippocratic) randomness** is ML randomness (for non-computable measures) which doesn’t use the measure as an oracle.

Theorem (R.)

*If \( \rho \) is computable (resp. left c.e.), the following are equivalent.*

- \( x \) is random (resp. blind random) on \( C \).
- \( x \) is random on \( \rho \) using Bienvenu/Hölzl/Porter/Shafer definitions.
- \( x \) is blind \( \mu \)-random for every probability measure \( \mu \geq \rho \).
There is so much more to say...

...but I have run out of time.
Summary

Randomness for nonadditive measures...
- provides a unified framework for concepts in randomness
- helps us to prove new results
- simplifies the proofs of old results
- gives us 60-years-worth of theorems in capacity theory to draw from!
Closing Thoughts
Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, “Jason Rute”.