Math 523: Written Assignment 1

1. (a) Generalize the argument that led to Newton’s method to derive the cubically convergent iteration:

\[ x_{n+1} = x_n - \frac{1}{2} \left[ \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} \right]. \]

Hint: Consider \( 0 = f(x_n + \Delta) \approx f(x_n) + \Delta f'(x_n) + \frac{\Delta^2 f''(x_n)}{2} \) and solve for the first \( \Delta \) in the RHS to get

\[ \Delta \approx - \frac{f(x_n)}{f'(x_n)} + \frac{\Delta^2 f''(x_n)}{2 f'(x_n)}. \]

By setting \( \Delta = 0 \) in the RHS, we get in the LHS the Newton approximation. Plugging this into the RHS gives the desired result.

(b) We can verify the quadratic convergence of the Newton iteration using one line of Mathematica as follows: Use the Mathematica command

\[ x_1 = \text{Series}[x_0 - (f[x_0]/f'[x_0]), \{x_0, 0, 2\}]/.f[0] \to 0. \]

Here we have simplified the algebra by assuming that the root of \( f(x) \) is zero, which is not a restriction since Newton’s method is invariant to translations in the \( x \)-direction. Your task is to similarly verify the iteration in part a. is indeed cubically convergent using a similar Mathematica command.

Solution

(a) We wish to compute a root, \( x = \tilde{x} \), of the function \( f(x) \) using the given iteration with some initial guess \( x_n \). Consider

\[ x_{n+1} = x_n + \Delta x. \]

We want to solve

\[ f(x_n + \Delta x) = 0. \]

Using a Taylor expansion about \( x_n \) we have

\[ 0 = f(x_n) + \Delta x f'(x_n) + \frac{(\Delta x)^2}{2} f''(x_n) + O((\Delta x)^3). \]

Dropping the \( O((\Delta x)^3) \) terms assuming \( \Delta x \) sufficiently small and solving for \( \Delta x \) gives

\[ \Delta x \sim - \frac{f(x_n)}{f'(x_n)} + \frac{(\Delta x)^2 f''(x_n)}{2 f'(x_n)} \]

or dropping the second order terms

\[ \Delta x \sim - \frac{f(x_n)}{f'(x_n)} \quad (1) \]

Now just plugging (2) into (1) gives the desired result.

(b) Use the mathematica commands

\[ x_{n+1} = \text{Series}[x_n - (f[x_n]/f'[x_n]), \{x_n, 0, 2\}]/.f[0] \to 0 \]

and

\[ x_{n+1} = \text{Series}[x_n - (f[x_n]/f'[x_n]) - ((f[x_n])^2 * f''[x_n])/(2 * (f'[x_n]))^3, \{x_n, 0, 3\}]/.f[0] \to 0 \]

2. We showed in class how to derive an estimate of the convergence rate (given by order of convergence \( p \)) for the Secant method using Newton Divided Differences. Since \( (x_1, f(x_1)), (x_2, f(x_2)), (x_3, 0) \) lie on a straight line, for \( x_3 \) computed in terms of \( x_1 \) and \( x_2 \) using the Secant method, the final result was of the form

\[ x_3 = cx_1 x_2, \]

where \( c \) is a constant.
where here again we assume for simplicity that the root is zero, so that this equation gives a relation for the error. From this point on it is straightforward to show that for the Secant method

\[ e_{n+1} = c \cdot e_n^p, \quad p = \frac{1 + \sqrt{5}}{2}. \]

Use the same Newton Divided Difference approach to show that for Muller’s method

\[ x_4 = xx_1x_2x_3. \]

From this result, a convergence estimate such as the one above for Secant method but now with \( p \approx 1.84 \) holds. Note that you do not have to show the latter result.

Solution

It suffices to consider

\[ f(x) = \alpha x - \beta x^2 + \gamma x^3 \]

as we iterate to find the root \( x = 0 \). As in the example for the Secant method the four points \((x_1, f_1), (x_2, f_2), (x_3, f_3), \) and \((x_4, 0)\) lie on the parabola. Consider the divided difference table

Muller Method

\[
\begin{array}{cccc}
 x_1 & f_1 & f_2 & f_3 \\
 x_2 & - & f_2 - f_1 & f_3 - f_2 \\
 x_3 & & - & f_3 - f_2 \\
 x_4 & & & -f_3 \\
\end{array}
\]

The rightmost entry must be zero, thus we set the numerator to zero:

\[
\gamma x_1x_2x_3 + \alpha x_4 - \gamma x_1x_2x_4 - \gamma x_2x_3x_4 - \gamma x_2x_3x_4 + \beta x_4^2 + \gamma x_1x_4^2 + \gamma x_2x_4^2 + \gamma x_3x_4^2 = 0
\]

and as we converge to the root the \( \gamma x_1x_2x_3 \) and \( \alpha x_4 \) terms in the above expression dominate giving

\[ x_4 \approx \frac{-\gamma}{\alpha} x_1x_2x_3 \]

which yields the desired result.

3. In class we showed the following result for the Fixed Point Iteration \( x = g(x) \): Assume \( g(x) \in \mathbb{C}([a, b]), g([a, b]) \subset [a, b], \) and

\[ \lambda = \max_{a \leq x \leq b} |g'(x)| < 1. \]

Then \( \forall x_0 \in [a, b], x_{n+1} = g(x_n), n \geq 0, \) has a unique fixed point \( \bar{x} \in [a, b], \)

\[ \lim_{n \to \infty} x_n = \bar{x}, \]

and

\[ \lim_{n \to \infty} \frac{\bar{x} - x_{n+1}}{\bar{x} - x_n} = g'(\bar{x}). \]

More generally, the order of convergence is defined as follows. If \( g([a, b]) \in \mathbb{C}^p([a, b]), \) \( p \geq 2, \) and

\[ g'(\bar{x}) = \cdots = g^{p-1}(\bar{x}) = 0 \quad \text{and} \quad g^p(\bar{x}) \neq 0, \]

then for \( x_0 \) sufficiently close to \( \bar{x} \)

\[ \lim_{n \to \infty} \left| \frac{\bar{x} - x_{n+1}}{(\bar{x} - x_n)^p} \right| = \left| \frac{g^{(p)}(\bar{x})}{p!} \right|. \]
(a) Verify the latter general result in (3) under the assumptions \( g([a, b]) \in C^p([a, b]), p \geq 2, \) and
\[
g'(\tilde{x}) = \cdots = g^{p-1}(\tilde{x}) = 0 \quad \text{and} \quad g^p(\tilde{x}) \neq 0.
\]

**Solution:** Use a Taylor expansion of \( g(x) \).

(b) Consider instead the FPI:
\[
x_{n+1} = g(x_n) = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n}.
\]
Show that if \( \tilde{x} \) is a fixed point of \( g(x) \) then it is a fixed point of \( G(x) \).

**Solution:** \( \tilde{x} = g(\tilde{x}) \Rightarrow \tilde{x} = G(\tilde{x}) \)
\[
x = x - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x} = \frac{g(g(x)) + xg'(g(x))g'(x) - 2g(x)g'(x)}{g'(g(x))g'(x) - 2g'(x) + 1}
\]
by L’Hopitals, which equals \( \tilde{x} \) for \( x = \tilde{x} \), if \( g'(\tilde{x}) \neq 1 \).

(c) Consider the iterative function \( g(x) = x^2 \) and deduce convergence properties for both fixed point methods around the roots \( x = 0 \) and \( x = 1 \).

**Solution:** Let \( x = g(x) = x^2 \Rightarrow x = G(x) = \frac{x^4 - x^4}{x^4 - 2x^4 + x^2} = \frac{x^3}{2x^3 + x^3} \).
At \( x = 0 \)
\[
g(0) = g'(0) = 0 \quad \text{and} \quad g''(0) = 2
\]
giving quadratic convergence. Similarly
\[
G(0) = G'(0) = G''(0) \quad \text{and} \quad G'''(0) = 3
\]
giving a cubically convergent iteration. Note that \( g(g(x)) = x^4 \) giving a quartically convergent method, implying this method converges with fewer number of evaluations of \( g(x) \) to converge than using \( G(x) \).
At \( x = 1 \) \( g'(1) = 2 \) implying divergence. However,
\[
G(1) = G'(1) = 0, \quad \text{and} \quad G''(1) \neq 0
\]
giving quadratic convergence.