Hirt’s Stability Analysis

This method was originally published by C.W. Hirt as "Heuristic Stability Theory for Finite-Difference Equation," in Volume 2 of the Journal of Computational Physics, pp. 339-355 (1968). To illustrate the basics of this approach I'll use the same simple advection equation used to demonstrate a von Neumann (Fourier) stability analysis.

\[ \frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial x} = 0 \]

Again we will look at an "upwind" difference method and assume that \( V > 0 \).

\[ \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + V \frac{\rho_i^n - \rho_{i-1}^n}{\Delta x} = 0 \]

We start with a standard error analysis. Expand terms about the point \( i \) in space and \( n \) in time.

\[ \rho_i^{n+1} = \rho_i^n + \Delta t \frac{\partial \rho}{\partial t} \bigg|_i + \frac{\Delta t^2}{2} \frac{\partial^2 \rho}{\partial t^2} \bigg|_i + \frac{\Delta t^3}{6} \frac{\partial^3 \rho}{\partial t^3} \bigg|_i + \ldots \]

\[ \rho_i^n = \rho_i^n - \Delta x \frac{\partial \rho}{\partial x} \bigg|_i + \frac{\Delta x^2}{2} \frac{\partial^2 \rho}{\partial x^2} \bigg|_i - \frac{\Delta x^3}{6} \frac{\partial^3 \rho}{\partial x^3} \bigg|_i + \ldots \]

Substituting these Taylor expansions into the difference equation gives:

\[ \frac{\partial \rho}{\partial t} \bigg|_i + V \frac{\partial \rho}{\partial x} \bigg|_i + \frac{\Delta t^2}{2} \frac{\partial^2 \rho}{\partial t^2} \bigg|_i - V \frac{\Delta x^2}{2} \frac{\partial^2 \rho}{\partial x^2} \bigg|_i + \ldots = 0 \]

At this point we can note that the approximation is first order accurate in space and time. However there is more to be learned. The first spatial error term looks a lot like a diffusion term. To push this a little further, remember that the original advection equation gives:

\[ \frac{\partial \rho}{\partial t} = -V \frac{\partial \rho}{\partial x} \]

Taking the derivative with respect to time of this equation gives:

\[ \frac{\partial^2 \rho}{\partial t^2} = -V \frac{\partial^2 \rho}{\partial t \partial x} = -V \frac{\partial^2 \rho}{\partial x \partial t} = -V \frac{\partial}{\partial x} \bigg| \frac{\partial \rho}{\partial t} \bigg| \]

Substituting in again the expression for time derivative of density gives:
\[
\frac{\partial^2 \rho}{\partial t^2} = -V \frac{\partial \rho}{\partial x} \quad \text{or} \quad \frac{\partial^2 \rho}{\partial t^2} = -V \frac{\partial \rho}{\partial x} \left( -V \frac{\partial \rho}{\partial x} \right) = V^2 \frac{\partial^2 \rho}{\partial x^2}
\]

This means that the results of the error analysis can be written as:

\[
\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial x} + V^2 \Delta t \frac{\partial^2 \rho}{\partial x^2} - V \Delta x \frac{\partial^2 \rho}{\partial x^2} + \ldots = 0
\]

or

\[
\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial x} - \frac{V}{2} (\Delta x - V \Delta t) \frac{\partial^2 \rho}{\partial x^2} + \ldots = 0
\]

or

\[
\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial x} = \frac{V}{2} (\Delta x - V \Delta t) \frac{\partial^2 \rho}{\partial x^2} + O(\Delta x^2, \Delta t^2)
\]

This is an advection-diffusion equation with diffusion coefficient

\[
D = \frac{V}{2} (V \Delta x - \Delta t)
\]

Think about solution of the standard diffusion equation, using separation of variables. You will discover that when D<0, solutions to the pure diffusion equation grow exponentially with time. To have bounded, physical solutions we need \(\Delta t \leq \Delta x/V\).