1. (25 points) At time $t = 0$, a satellite has the following state

$$\mathbf{r}_0 = -5529\hat{i} - 19030\hat{j} + 0\hat{k} \text{ (km)}$$

$$\mathbf{v}_0 = 4.60\hat{i} + 3.07\hat{j} - 0.94\hat{k} \text{ (km/s)}$$

where $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors defining the Earth-centered inertial frame ($\hat{i}$ points toward Aries, and $\hat{k}$ is aligned with Earth’s spin axis).

a) (8 pts) Calculate the inclination of this orbit.

$$i = \cos^{-1}\left(\frac{\mathbf{h} \cdot \hat{k}}{h}\right)$$

$$\mathbf{h} = \mathbf{r}_0 \times \mathbf{v}_0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5529 & -19030 & 0 \\ 4.60 & 3.07 & -0.94 \end{vmatrix} = 17888.2\hat{i} - 5197.3\hat{j} + 70563.9\hat{k} \text{ (km$^2$/s)}$$

$$h = 72981.3 \text{ km/s}$$

$$i = \cos^{-1}\left(\frac{70563.9}{72981.3}\right) = \cos^{-1}(0.96187) = 14.8 \text{ deg}$$

$$\mathbf{h} = 0.258 \text{ rad}$$

b) (8 pts) Calculate the eccentricity vector for this orbit.

$$\mathbf{e} = \frac{1}{\mu} (\mathbf{v} \times \mathbf{h}) - \frac{\mathbf{r}}{r}$$

$$\mathbf{v} \times \mathbf{h} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4.60 & 3.07 & -0.94 \\ 17888.2 & -5197.3 & 70563.9 \end{vmatrix} = 211745.7\hat{i} - 341408.8\hat{j} - 78824.4\hat{k} \text{ (km$^3$/s)}$$

$$\mathbf{e} = \frac{1}{3.986 \times 10^{14} \text{ km}^3/\text{s}^2} \left(\mathbf{v} \times \mathbf{h}\right) - \frac{\mathbf{r}}{r}$$

$$\mathbf{e} = 0.810\hat{i} + 0.1038\hat{j} - 0.1977\hat{k}$$

(continued on next page)
c) (9 pts) Calculate the satellite's true anomaly at $t = 0$.

\[ e = |\bar{e}| = 0.84 \]

\[ r_0 = \frac{p}{1 + e \cos \Theta_0} \]

\[ p = a(1-e^2) \]

\[ a = \frac{-\mu}{2 \epsilon} \]

\[ \epsilon = \frac{\sqrt{\gamma}}{2} - \frac{\mu}{2a} \]

\[ v_0 = |\bar{v}_0| = 5.61 \text{ km/s} \]

\[ r_0 = 19816.9 \text{ km} \]

\[ \rightarrow \epsilon = -4.378 \text{ km}^2/\text{s}^2 \]

\[ \rightarrow a = \frac{-\mu}{2(-4.378 \text{ km}^2/\text{s}^2)} = 45.523 \]

\[ p = 13401 \text{ km} \]

\[ r_0 = \frac{p}{1 + e \cos \Theta_0} \]

\[ \cos \Theta_0 = \frac{1}{e} \left( \frac{p}{r_0} - 1 \right) \]

\[ = 0.3854 \]

\[ \Theta_0 = \pm 112.7 \text{ deg} \]

Check quadrant $\rightarrow \bar{r}_0 \cdot \bar{v}_0 = -83855 \text{ km}^2/\text{s}^2$

$\Theta_0$ in range $180^\circ - 360^\circ$

$\therefore \Theta_0 = -112.7 \text{ deg}$

(geometrical equivalent to $360^\circ - 112.7^\circ$

$= 247.3^\circ$)
2. (25 points) A s/c on a low Earth-orbit, initially circular, is experiencing atmospheric drag, which results in a perturbing acceleration given by

\[ \ddot{a}_p = k v^2 \left( -\frac{\ddot{v}}{v} \right) = -k v \ddot{v} \]

where \( k \) is a positive constant, \( \ddot{v} \) is the vehicle's velocity and \( v \) is its magnitude.

a.) (15 pts) Convert this acceleration into a form suitable for use in Gauss' variational equations (i.e., in terms of orbital elements and true anomaly).

\[ \ddot{r} = \frac{1}{\mu} \left[ \frac{\ddot{v}}{1+e \cos \theta} \right] = \frac{-\ddot{v} (-e \dot{v} \sin \theta)}{(1+e \cos \theta)^2} = \frac{pe \dot{v} \sin \theta}{(1+e \cos \theta)^2} \]

\[ r \dot{\theta} = \frac{h}{\mu} = \frac{\sqrt{\mu} \sqrt{e \sin \theta}}{(1+e \cos \theta)} = \frac{\mu}{\sqrt{\mu} \sqrt{e}} \sqrt{(1+e \cos \theta)} \]

\[ \ddot{a}_p = -k v \ddot{v} = -k V \left[ \frac{-\mu}{\sqrt{\mu} \sqrt{e}} \sqrt{e \sin \theta} \ddot{r} + \sqrt{\frac{\mu}{\sqrt{\mu} \sqrt{e}}} \left( 1+e \cos \theta \right) \ddot{\theta} \right] \]

b.) (12 pts) Convert the differential equations with

\[ V = \sqrt{\frac{\mu}{\sqrt{\mu} \sqrt{e}}} \sqrt{e \sin \theta} \sqrt{(1+e \cos \theta)} \]

into state variable form for use in numerical integration (i.e., a set of first-order differential equations with no differentiation occurring on the right-hand side). The quantities \( \dot{x}_i \) and \( \ddot{x}_i \) are the components of the atmospheric perturbation given at the top of the page. Be sure to convert them into state-variable form as well.

\[ \ddot{a}_p = -k \sqrt{\dot{x}^2 + \dot{y}^2} (\dot{x} \ddot{x} + \dot{y} \ddot{y}) \]

\[ \dot{x}_i = \dot{x}_i \]

\[ \dot{X}_i = \dot{X}_i \]

\[ \dot{X}_4 = \dot{X}_3 \]

\[ \dot{X}_3 = \frac{-\mu x_1}{\sqrt{X_1^2 + X_2^2}} - k \sqrt{X_1^2 + X_2^2} \frac{x_2}{x_2} \]

\[ \dot{X}_2 = \frac{-\mu x_3}{\sqrt{X_1^2 + X_2^2}} - k \sqrt{X_1^2 + X_2^2} \frac{x_3}{x_3} \]
3. (25 points) A s/c is using ion thrusters (low thrust but high efficiency) to change some orbital elements.

   a) (8 pts) In which direction should the thrust be pointed to achieve maximum positive rate of change of semi-major axis if the thruster is turned on only near periapsis? Explain your answer using Gauss’ variational equations.

   Near periapsis, \( \sin \theta \approx 0 \), so only the term \((1+e \cos \theta)\dot{a}\) will be significant.

   Therefore, max. change occurs if \( \ddot{a} = a \hat{n} \)

   So, direct thrust along \( \hat{e} \).

   b) (8 pts) In which direction should the thrust be pointed to achieve maximum positive rate of change in inclination for any value of true anomaly? Explain your answer using Gauss’ variational equations.

   Inclination changes only if \( \ddot{a} \) has component along \( \hat{e}_z \), therefore that is the desired direction.

   Note that if \( \theta + \omega > 180^\circ \), then need to direct thrust along \( -\hat{e}_z \).

   b) (9 pts) Under what conditions could an Earth-orbit be sun-synchronous and have its periapsis fixed in the orbital plane? Would this be a realistic orbit?

   1. Node must progress at rate \( \frac{d\omega}{dt} = \frac{2\pi}{\text{year}} = 1.997 \times 10^{-6} \text{ rad/s} \)

      This requires \( 90^\circ < \iota < 150^\circ \)

   2. Periapsis must not drift in orb. plane

      \( \frac{d\omega}{dt} = 0 \rightarrow 1 - 5 \cos^2 \iota = 0 \)

      \( \cos^2 \iota = \frac{1}{5} \)

      \( \iota = 63.4^\circ \text{ or } 116.5^\circ \)

      Could also use \( r_g \) = \( a(1-e) \)

      To eliminate from this eqn. and set \( \rho = 6679 \text{ km} \)

      and solve for \( a \).

      Use eqn. \( 1.997 \times 10^{-7} s^{-1} \rightarrow \left( 1 - \frac{1}{\rho^2} \right) \cos \iota \left( \iota / 2 \right) = \frac{\left( 1+e \cos \theta \right)}{1+\cos \iota} \)

      Try \( e = 0 \rightarrow \iota = 71.5^\circ \)

      \( a = 1.078 \times 10^{14} \)

      \( \rho = 9800 \text{ km.} \) --- realistic

      \( a = 9800 \text{ km.} \) --- realistic
4. (25 points) A satellite on an orbit with $a = 25000$ km and $e = 0.7$ passes through periapsis at $t = 0$.

Use an iterative method to estimate the satellite’s true anomaly $\theta$ at $t = 12500$ s.

You must show your work. Carry out the iteration for just one iterative step.

$$n = \sqrt{\frac{\mu}{a^3}} = 1.597 \times 10^{-4} \text{ s}^{-1}$$

$$M = n(t - t_p) = n(12500 \text{ s} - 0 \text{ s}) = 1.996$$

$$M = E - e \sin E$$

Using fixed-pt. iteration:

$$E_{k+1} = M + e \sin E_k$$

$$E_2 = 1.996 + 0.7 \sin E_1$$

Guess $E_1 = \frac{M}{1-e} = 6.653 \text{ rad}$

$$E_2 = 2.244 \text{ rad}$$

$$E_3 = M + e \sin E_2$$

$$= 2.541$$

$$\frac{\pi}{2} - 2.541 \text{ rad.}$$

$$E_4 = M + e \sin E_3$$

$$= 2.391 \text{ rad.}$$

Convert $E_2$ to $\theta$:

$$\theta = 2 \tan^{-1}\left[ \sqrt{\frac{1+e}{1-e} \sin \frac{E_2}{2}} \right]$$

$$= 157.26 \text{ deg. (}= 2.744 \text{ rad.})$$

Using this method, no need to check quadrant.

Since $\frac{E}{2}$ and $\frac{\theta}{2}$ will lie in same quadrant.