(20 points) Consider the following rendezvous problem: a target satellite is in a circular orbit with \( a = 7500 \) km. The Space Shuttle must rendezvous with the target at \( t_f = 15000 \) s. The initial relative position and speed of the Shuttle are

\[
\begin{bmatrix}
-4 \\
-50 \\
5
\end{bmatrix} \text{ (km)} \quad \begin{bmatrix}
0.02 \\
0.07 \\
-0.1
\end{bmatrix} \text{ (km/s)}
\]

Calculate the

a.) required initial velocity \( \dot{\mathbf{r}}(0) \) to initiate the rendezvous maneuver.
b.) \( \Delta \dot{v} \) needed to initiate the rendezvous maneuver.

c.) relative velocity of \textit{arrival} at the target \( \dot{\mathbf{r}}(t_f) \)
d.) \( \Delta \dot{v} \) needed to complete the rendezvous, i.e., to achieve zero relative velocity.

\[
\mathbf{S}_{0}\,^{/n}:
\]

a.) \( \mathbf{\dot{r}}(t_f) = \mathbf{M}(t_f) \mathbf{\dot{r}}(0) + \mathbf{N}(t_f) \mathbf{\dot{r}}(0) \)

But at \( t_f \), \( \mathbf{\dot{r}}(t_f) = 0 \)

\( \mathbf{\dot{r}}(0) \)

Sign indicates \textit{after} impulse occurs.

\( \mathbf{0} = \mathbf{M}(t_f) \mathbf{\dot{r}}(0) + \mathbf{N}(t_f) \mathbf{\dot{r}}(0) \)

\( \mathbf{N}(t_f) = -\mathbf{M}(t_f)^{-1} \mathbf{M}(t_f) \mathbf{\dot{r}}(0) \)

First, evaluate \( \mathbf{M}(t_f), \mathbf{N}(t_f), \) and solve \( \mathbf{N}(t_f)^{-1} \)

\( \mathbf{N}(t_f) = \begin{bmatrix} 929 & 2940 & 0 \\ -2940 & 41282 & 0 \\ 0 & 0 & 929 \end{bmatrix} \)

\( \mathbf{N}(t_f)^{-1} = \begin{bmatrix} 0.0014 & 0.0001 & 0 \\ -0.0001 & 0 & 0 \\ 0 & 0 & 0.0011 \end{bmatrix} \) (calculated using Matlab)
\[ \dot{\mathbf{v}}(0)^+ = -N(t_f)^{-1}M(t_f)\mathbf{v}(0) = \begin{bmatrix} 0.0018 \\ 0.0066 \\ 0.0023 \end{bmatrix} \text{ (km/s)} \]

b.) \[ \Delta \mathbf{v}_o = \dot{\mathbf{v}}(0)^+ - \dot{\mathbf{v}}(0)^- = \mathbf{v}(0)^- = \text{rel. vel. before imp} \]
\[ = \begin{bmatrix} 0.0018 \\ 0.0066 \\ 0.0023 \end{bmatrix} \text{ (km/h)} - \begin{bmatrix} 0.02 \\ 0.07 \\ -0.10 \end{bmatrix} = \begin{bmatrix} -0.0182 \\ -0.0634 \\ 0.1023 \end{bmatrix} \text{ (km/s)} \]

\[ \Delta \mathbf{v}_o = \begin{bmatrix} 0.02 \\ 0.07 \\ -0.10 \end{bmatrix} \text{ (km/s)} \]

\[ \mathbf{v}(0)^- = \begin{bmatrix} 0.02 \\ 0.07 \\ -0.10 \end{bmatrix} \text{ (km/s)} \]

c.) Use soln. to the C-W eqs. to calc. \( \mathbf{v}(t) \) at \( t = t_f \)
\[ \mathbf{v}(t) = \Sigma(t)\mathbf{v}(0) + \mathbf{I}(t)\dot{\mathbf{v}}(0) \]
\[ \dot{\mathbf{v}}(t_f) = \Sigma(t_f)\mathbf{v}(0) + \mathbf{I}(t_f)\dot{\mathbf{v}}(0) \]
\[ \Sigma(t_f) = \begin{bmatrix} 3.04 & 0 & 0 \\ 0 & -0.276 & 0 \\ 0 & 0 & -ns \end{bmatrix} \begin{bmatrix} 0.0024 & 0 & 0 \\ -0.0083 & 0 & 0 \\ 0 & 0 & -0.0069 \end{bmatrix} \]
\[ \mathbf{I}(t_f) = \begin{bmatrix} c & 2.5 & 0 \\ -2.5 & 4c - 3 & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} -0.4288 & 1.8068 & 0 \\ -1.8068 & -4.7153 & 0 \\ 0 & 0 & -0.4288 \end{bmatrix} \]
\[ \dot{\mathbf{v}}(t_f) = \begin{bmatrix} 0.0066 \\ -0.0012 \\ -0.0054 \end{bmatrix} \text{ (km/s)} \]

\[ \dot{\mathbf{v}}(t_f)^- = \dot{\mathbf{v}} \text{ before the second impulse is applied.} \]

\[ \dot{\mathbf{v}}(t_f)^- = \begin{bmatrix} 0.0066 \\ -0.0012 \\ -0.0054 \end{bmatrix} \text{ (km/s)} \]

\( \dot{\mathbf{v}}(t_f)^+ \text{ must use } \dot{\mathbf{v}}(t_f)^+ \text{ since that is the rel. vel. after the first impulse that initiated the rendezvous trajectory.} \)
d.) To complete the rendezvous, need to achieve \( \dot{\vec{r}}(t_f)^+ = 0 \)

That is, after the second impulse \( (t_f) \), rel vel. med. = 0.

\[
\Delta v_f = \dot{\vec{r}}(t_f)^+ - \dot{\vec{r}}(t_f)^-
\]

\[
= 0 - \dot{\vec{r}}(t_f)^-
\]

\[
\Delta v_f = \begin{bmatrix} -0.0006 \\ 0.0012 \\ 0.0054 \end{bmatrix} \text{ (km/s)}
\]
(15 points) A satellite must change its orbit to a circular orbit with radius $r_0 = 7800$ km. The satellite is presently in an elliptical orbit with $a = 9000$ km and $e = 0.24$.

a.) Determine the true anomaly where the $\Delta v$ should occur.
b.) How many seconds after periapsis passage does the $\Delta v$ occur?
c.) Calculate the magnitude of the $\Delta v$ and its direction relative to the local horizontal plane at that point. (Draw a picture defining this angle!)

Solu:

a.) elliptical (initial) orbit has $p = a(1-e^2) = 8481.6$ km

new orbit (circular) intersects old orbit at $r_0 = 7800$ km.

$$7800 \text{ km} = \frac{r}{1+e\cos\theta}$$

$$\theta = \cos^{-1}\left[ \frac{1}{e} \left( \frac{p}{r_0} - 1 \right) \right]$$

$$\theta = \pm 68.65 \text{ deg}$$

Either location will require same $\Delta v$ because of symmetry.

b.) Use Kepler's eqn to calc. time after periapsis passage.

$$n(t-t_p) = E - e \sin E$$

$$E = 2 \tan^{-1}\left[ \sqrt{1-e} \tan \frac{\theta}{2} \right]$$

$$= 0.9818 \text{ rad}.$$ 

$$n = \sqrt{\frac{\mu}{a^3}} = 7.3444 \times 10^{-4}$$

$$\rightarrow t-t_p = 1057.9 \text{ sec.}$$
c.)

Initial orbit: $V_1$ (at $\theta = 68.65\,\text{deg}$)

$$V_1 = \sqrt{2(\varepsilon_1 + \frac{\mu}{R_0})},$$

$$\varepsilon_1 = -\frac{\mu}{2a_1} = -22.144\,\text{km/s}^2$$

$\rightarrow V_1 = 7.6103\,\text{km/s}$.

Flight-pole angle $\beta_1 = \cos^{-1}\left(\frac{h}{R_0 V_1}\right) = \cos^{-1}\left(\frac{\mu R_0}{\mu R_0 V_1}\right)$

$\beta_1 = 11.6\,\text{deg}$.

Circular orbit will have

$$V_2 = \sqrt{\frac{\mu}{R_0}} = 7.1486$$

and flight-pole angle $\beta_2 = 0$ (circular orbit)

$\Delta\beta = \beta_2 - \beta_1 = -11.6\,\text{deg}$.

Calc. $\Delta V$ using law of cosines

$$\Delta V = \sqrt{V_1^2 + V_2^2 - 2V_1 V_2 \cos(\Delta\beta)}$$

$\Delta V = 1.5606\,\text{km/s}$
Can specify direction of $\Delta \vec{v}$ wrt local horiz. by using either angles $\gamma$ or $\lambda$

Law of sines gives \[
\frac{\sin \beta_1}{\Delta v} = \frac{\sin \gamma}{v_1}
\]
\[
\gamma = \sin^{-1}(0.9806)
\]
\[
\gamma = 78.6 \text{ deg} \quad \text{or} \quad 101.4 \text{ deg.}
\]
Cannot determine which is correct!

Better use law of cosines here:

\[
v_1^2 = v_2^2 + (\Delta v)^2 - 2 v_2 \Delta v \cos \gamma
\]
\[
\cos \gamma = -0.1912
\]
\[
\gamma = 101.4 \text{ deg.}
\]

(or can specify $\lambda$, given by $\lambda = 180 - \gamma = 78.6 \text{ deg.}$)
(20 points) A lunar flight was initially launched into an orbit with flight-path angle \( \beta_0 = 20 \) degrees, \( v_0 = 8.28 \) km/s, and \( r_0 = 7000 \) km.

a.) Calculate the initial true anomaly \( \theta_0 \) and the eccentricity \( e_0 \).

b.) After exactly one rev the spacecraft was injected into a translunar trajectory without changing the line of apsides. Determine \( \beta \) and \( v \) immediately after applying the impulse if the new apogee is at the mean lunar distance \( (3.844 \times 10^3) \) km.

**Solu:**

\[ h = r_0 v_0 \cos \beta_0 = 5446.58 \text{ km/s} \]
\[ p = h^2 \mu = 7442 \text{ km}. \]
\[ \epsilon = \frac{v_0^2}{2} - \frac{\mu}{r_0} = -22.464 \text{ km/s}^2 \]
\[ \rightarrow a = \frac{2}{\epsilon} = 8793.68 \text{ km}. \]

Must first calculate \( e_0 \) before calculating \( \theta_0 \).

\[ a = \frac{p}{1 - e_0^2} \]
\[ e_0^2 = 1 - \frac{a}{p} \]
\[ e_0 = \sqrt{1 - \frac{a}{p}} = 0.392 \]

\[ r_0 = \frac{p}{1 + e_0 \cos \theta_0} \rightarrow \theta_0 = \cos^{-1} \left[ \frac{1}{e_0} \left( \frac{p}{r_0} - 1 \right) \right] \]
\[ = \pm 80.7 \text{ deg.} \]

Since \( \beta_0 > 0 \), then \( \theta_0 \) lies in range \( 0 < \theta < 180^\circ \).

\[ \theta_0 = +80.7 \text{ deg.} \]

b.) The \( \Delta v \) will put us on a new orbit with the same line of apsides \( \rightarrow \) true anomaly will be the same immediately before and after the \( \Delta v \). Also, the radius stays the same immediately before and after \( \Delta v \).
\[ r_o = \frac{p_1}{1 + e_1 \cos \theta_1} = \frac{p_0}{1 + e_0 \cos \theta_0}. \]

But \( \theta_1 = \theta_0 = 80.7 \text{ deg} \).

\[ \frac{p_1}{1 + 0.1616 e_1} = 6998.6 \text{ km}. \]

\[ p_1 = 6998.6 \text{ km} + (1130.97 \text{ km}) e_1 \] \hspace{1cm} (1)

Need another eqn involving \( p_1 \) and \( e_1 \).

\( \rightarrow \) Use the condition that apogee of new orbit has:

\( r_a = \text{mean lunar distance} = 3.844 \times 10^5 \text{ km} \).

\[ r_a = \frac{p_1}{1 + e_1 \cos \theta_a} \]

\[ \theta_a = 180^\circ \]

\[ r_a = 3.844 \times 10^5 \text{ km} = \frac{p_1}{1 - e_1} \]

\[ p_1 = 3.844 \times 10^5 \text{ km} - (3.844 \times 10^5 \text{ km}) e_1 \] \hspace{1cm} (2)

Subst. \( p_1 \) from Eq (2) into Eq (1):

\[ 3.844 \times 10^5 \text{ km} - (3.844 \times 10^5 \text{ km}) e_1 = 6998.6 \text{ km} + (1130.97 \text{ km}) e_1 \]

\[ \rightarrow e_1 = 0.9787 \] (nearly a parabola!)

Can now calc. perigee radius:

\[ r_p = \frac{p_1}{1 + e_1} = \frac{810.8 \text{ km}}{1 + 0.9787} \]

\[ r_p = 4098.66 \text{ km} \]

\[ a_1 = \frac{r_a + r_p}{2} = \frac{3.844 \times 10^5 \text{ km} + 4098.66 \text{ km}}{2} \]

\[ e_1 = \frac{\mu}{2a_1} = -1.02 \text{ km/s}^2 \]
\[ V_1 = \sqrt{2 \left( \frac{\varepsilon}{c^2} + \frac{1}{c^2} \right)} = 10.575 \text{ km/s} \]

New \( h_i = v_i \cos \beta_i = \sqrt{\mu r_i} = \sqrt{\mu \cdot 8110.8 \text{ km}^3/\text{s}^2} = 56859.17 \text{ km/s} \)

\[ \cos \beta_i = \frac{h_i}{v_i} = \frac{h_i}{r_i v_i} = 0.7681 \]

\[ \beta_i = 39.8 \text{ deg.} \]

(Select + value, since \( \Theta_i = +80.7 \text{ deg.} \))

\[ \Theta_i = 39.8 \text{ deg.} \]

(15 points) This problem compares the \( \Delta v \) for impulsive thrust with that of low-level, continual thrust used in HW B4.

a.) Calculate the total \( \Delta v \) for a Hohmann transfer from \( r_1 = 7000 \text{ km} \) to \( r_2 = 37600 \text{ km} \).

b.) In HW B4, you numerically integrated the equations for a s/c that used constant transverse thrust to spiral out from 7000 km to 37600 km. The total \( \Delta v \) (called the low-thrust \( \Delta v \)) for that problem is simply

\[ \Delta v_{LT} = \int_{r_1}^{r_2} a_t \cdot dt = a_t \cdot 10P \]

where \( P \) is the period of the initial circular orbit with radius \( r_1 = 7000 \text{ km} \), and \( a_t = 8 \times 10^{-5} \text{ km/s}^2 \).

Calculate \( \Delta v_{LT} \) and compare it with the \( \Delta v \) for a Hohmann transfer between circular orbits with radii of 7000 km and 37600 km. Why are these different?

c.) Calculate the transfer time for the Hohmann transfer in part a).

Solve:

a.) Hohmann xfer has \( a_H = \frac{\vec{r}_i + \vec{r}_f}{2} = 22300 \text{ km} \)

\[ \varepsilon_H = \frac{\mu}{2a_H} = -8.937 \text{ km}^2/\text{s}^2 \]
\[ V_p = \sqrt{2 \left( \frac{E_i}{m_i} + \frac{K_i}{m_i} \right)} \quad \text{but} \quad r_p = r_i \]
\[ = 9.7986 \text{ km/s} \]

\[ V_a = \sqrt{2 \left( \frac{E_a}{m_a} + \frac{K_a}{m_a} \right)} \quad \text{but} \quad r_a = r_a \]
\[ = 1.8243 \text{ km/s} \]

\[ V_i = \sqrt{\frac{\mu}{r_i}} = 7.5460 \text{ km/s} \quad \text{(vel on initial orbit)} \]

\[ V_e = \sqrt{\frac{\mu}{r_e}} = 3.2559 \text{ km/s} \quad \text{(vel on final orbit)} \]

\[ \Delta V_1 = V_p - V_i = 2.2526 \text{ km/s} \quad \text{(initial Hohmann xfer)} \]

\[ \Delta V_2 = V_e - V_a = 1.4344 \text{ km/s} \quad \text{(circular at geostationary orbit)} \]

\[ \Delta V_{\text{TOT}} \quad (H-xfer) = |\Delta V_1| + |\Delta V_2| = 3.6870 \text{ km/s} \]

b.) \[ \Delta V_{LT} = \int_0^{10P} a_T \, dt = a_T \cdot 10P \]

\[ P = 2\pi \sqrt{\frac{a_{3.0}}{\mu}} = 2\pi \sqrt{\frac{r_{3.0}^3}{\mu}} = 5928.5 \text{ sec} \]

\[ \rightarrow \Delta V_{LT} = 8 \times 10^{-5} \text{ km/s} \times 10 \times 5928.5 \text{ sec} \]

\[ \Delta V_{LT} = 4.7428 \text{ km/s} \]

This is larger than the \( \Delta V_{\text{TOT}} \) for a Hohmann xfer for 2 reasons: 1) gravity loss - the low-thrust method acts over a long time, whereas the H-xfer uses 2 impulses (which have no gravity losses); 2) The final orbit for the low-thrust method is almost, but not quite, circular. In this problem, reason #1 is the primary factor.
c.) xfer time = \( \frac{1}{2} \) period of Hohmann ellipse

\[ T_H = 2\pi \sqrt{\frac{a_H^3}{\mu}} = 33141.2 \text{ sec.} \]

\[ \implies \text{xfer time} = \frac{1}{2} T_H = 16570.6 \text{ sec} \]

\[ = 4.6 \text{ hrs.} \]