In these problems, you will need to use the concept of averaged rates. The average rate of change in an orbital element \( \alpha \) (denoted by angular brackets) is:

\[
\langle \frac{d\alpha}{dt} \rangle = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \langle \frac{d\alpha}{dt} \rangle_0 d\theta
\]

One can then approximate the change in \( \alpha \) over some time interval \( \Delta t = t_2 - t_1 \) (which corresponds to the angular change \( \theta_1 \) to \( \theta_2 \)) by using this averaged rate of change.

1. A satellite is using constant low-level thrust (generated by ion engines) to change its orbital radius. The initial orbit is circular with radius \( r_0 \). Using Gauss’ variational eqs., calculate the approximate change in \( a \) after one revolution if the thrust acceleration (thrust divided by satellite mass) \( a_T \) is always directed perpendicular to the radius vector \( F \) (and in the direction of the satellite’s motion). Assume that \( a \) and \( e \) change very little (since the thrust is small relative to the gravitational force). Ignore the effects of changes in the other orbital elements.

\[
\begin{align*}
\dot{r}_0 &= 7000 \text{ km,} \\
a_T &= 4.25 \times 10^{-7} \text{ km/s}^2
\end{align*}
\]

Solu. Perturbing accel is \( \ddot{a}_p = \hat{a}_r + a_T \hat{e}_\theta + 0 \hat{e}_z \)

\[
\begin{align*}
\frac{d\alpha}{dt} &= \frac{2}{n \sqrt{1 - e^2}} \left[ \int_{0}^{\theta} + (1 + e \cos \theta) a_T \right] \\
&= \frac{2a}{\sqrt{\mu (1 - e^2)}} \left[ (1 + e \cos \theta) a_T \right] \\
\langle \frac{d\alpha}{dt} \rangle_{\text{rev}} &= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{2a}{\sqrt{\mu (1 - e^2)}} \left[ (1 + e \cos \theta) a_T \right] d\theta \\
&= \frac{a}{\pi \sqrt{\mu (1 - e^2)}} \left[ \frac{2\pi}{3} \int_{0}^{2\pi} (1 + e \cos \theta) \right] d\theta \\
&= \frac{a}{\pi \sqrt{\mu (1 - e^2)}} \left[ \frac{2\pi}{3} \right] \\
&= \frac{a}{\pi \sqrt{\mu (1 - e^2)}} \left[ \frac{2\pi}{3} \right]
\end{align*}
\]
\[ \frac{\langle da \rangle}{dt} \text{ rev}^{-1} = \frac{2 \left( 7000 \text{ km} \right)^{3/2} \left( 4.25 \times 10^{-7} \text{ km/s}^2 \right)}{\sqrt{(3.986 \times 10^{5} \text{ km}^3/\text{s}^2) \left( 1 - e^2 \right)}} \]

\[ = 7.885 \times 10^{-4} \text{ km/s}. \]

\[ \Delta a \text{ over one rev} \propto \frac{\langle da \rangle}{dt} \text{ rev}^{-1} \times \text{ orb. period} \]

\[ T = 2\pi \sqrt{\frac{a^3}{\mu}} = 5828.5 \text{ sec.} \]

\[ \Delta a = 7.885 \times 10^{-4} \frac{\text{km}}{s} \times 5828.5 \text{ sec} \]

\[ \Delta a = 4.6 \text{ km} \]

2. Repeat Prob. 1, but instead of using Gauss’ variational eqs., estimate the change in a by calculating the increase in energy \( \Delta e \) created by the thrust acceleration. The rate of increase in \( \Delta e \) is

\[ \frac{de}{dt} = \bar{a}_T \cdot \bar{v} \]

where \( \bar{v} \) is the satellite’s velocity. (As usual, assume that \( a \) and \( e \) change very little over one rev.).

a.) From this result, calculate the change in energy over one rev., and hence the change in \( a \).

b.) Comment on the similarity of the answers in problems 1 and 2. Why would one expect this?

Solu:

a.) \[ \frac{de}{dt} = \bar{a}_T \cdot \bar{v} \]

\[ \text{speed on circular orbit with radius } a \]

\[ = a_T \hat{r}_0 \cdot \sqrt{\frac{\mu}{a}} \hat{r}_0 \]

\[ = a_T \sqrt{\frac{\mu}{a}} = 3.21 \times 10^{-6} \text{ km}^2/\text{s} \]

\[ \text{(this is approx. constant since } a \text{ changes slowly over one rev.)} \]

\[ \Delta e \text{ over one rev} \propto a_T \sqrt{\frac{\mu}{a}} \times T \]

\[ = a_T \sqrt{\frac{\mu}{a}} \times 2\pi \sqrt{\frac{a^3}{\mu}} = 2\pi a a_T = 0.1839 \text{ km}^2/\text{s}^2 \]
initial energy (before thrusting): $E_0 = \frac{-\mu}{2a_0} = -28.4714 \text{ km}^2 \text{s}^{-2}$

new energy (after one row of thrusting) $E_o + \Delta E$

$$= -28.4527$$

$$\Rightarrow a_{\text{new}} = \frac{-\mu}{2E_{\text{new}}} = \frac{7004.605 \text{ km}}{\Delta a = 4.605 \text{ km}}$$

b.) The answer are nearly identical since the variational eqn.
used in Prob 1 was derived using the concept of
energy change due to the perturbation.

Both approaches are only approximate though, and
+ be accurate, require that the perturbing accel. $a_T$
be small relative to the gravitational accel. $a_{\text{grav}}$. We can
easily verify that this is true for this problem:

$$a_{\text{grav}} = \frac{\mu}{a^2} = \frac{3.986 \times 10^5 \text{ km}^3 \text{s}^{-2}}{(7000 \text{ km})^3} = 0.008 \text{ km}^2 \text{s}^{-2}$$

$$a_T = 4.25 \times 10^{-2} \text{ km}^2 \text{s}^{-2} \ll a_{\text{grav}}$$
3. A perfectly spherical satellite, with a mirror-like surface is orbiting Earth on an elliptical path, with orbital elements $\alpha_0$ and $\epsilon_0$ (the remaining elements are not important in this problem). The orbital plane includes the Sun, and the line of apsides for the satellite's orbit also includes the Sun (see figure). Radiation pressure from sunlight perturbs this orbit with a constant force vector $\vec{F}_s$, resulting in a constant perturbing acceleration $\vec{a}_L$ parallel to the line of apsides (ignore times when the satellite passes into Earth's shadow).

a.) Calculate the approximate change in $a$ over one rev.
b.) Calculate the approximate change in $e$ over one rev.
c.) Calculate the approximate change in $a$ between periapsis and apoapsis.

First, find $a_r$ and $a_\theta$ components of $\vec{a}_p$.

$$ a_r = \vec{a}_L \cdot \hat{r}_r = a_L \cos \alpha_1 $$
$$ = a_L \cos(\pi - \theta) $$
$$ a_r = -a_L \cos \theta $$

$$ a_\theta = \vec{a}_L \cdot \hat{r}_\theta = a_L \cos \alpha_2 $$
$$ = a_L \cos(\pi - \theta + \frac{\pi}{2}) $$
$$ = a_L \sin \theta $$

$$ \vec{a}_p = -a_L \cos \theta \hat{r}_r + a_L \sin \theta \hat{r}_\theta $$
\[
\frac{da}{dt} = \frac{2 a^{3/2} a_L}{\sqrt{\mu(1-e^2)}} \left[ e \sin \theta (-\cos \theta) + (1 + e \cos \theta) \sin \theta \right]
\]
\[
= 2 a^{3/2} a_L \frac{\sin \theta}{\sqrt{\mu(1-e^2)}}
\]
\[
\langle \frac{da}{dt} \rangle_{\text{one rev}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{2 a^{3/2} a_L \sin \theta}{\sqrt{\mu(1-e^2)}} d\theta
\]
\[
= \frac{a^{3/2} a_L}{\pi \sqrt{\mu(1-e^2)}} \int_0^{2\pi} \sin \theta d\theta
\]
\[
= 0
\]

Therefore, \( e \) does not change on average over one rev.

b.) \( \frac{de}{dt} = \frac{\sqrt{1-e^2}}{n a} \left[ \sin \theta a_r + (\cos E + \cos \theta) a_y \right] \)

Convert \( \cos E \) into an expression involving \( \theta \).

By definition of \( E \), \( \cos E = \frac{e + \cos \theta}{1 + e \cos \theta} \)

\[-\frac{de}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}} \left[ -a_L \sin \theta \cos \theta + \left( \frac{e + \cos \theta}{1 + e \cos \theta} \right) a_L \sin \theta \right] \]

\[
\langle \frac{de}{dt} \rangle_{\text{one rev}} = \frac{e}{\pi} \int_0^{2\pi} \frac{e \sin \theta}{1 + e \cos \theta} d\theta + \frac{e}{\pi} \int_0^{2\pi} \frac{\cos \theta \sin \theta}{1 + e \cos \theta} d\theta
\]

since \( e \) is approximately constant, both integrands are odd functions — Therefore both integrals = 0.

\[
\langle \frac{de}{dt} \rangle_{\text{one rev}} = 0
\]

So \( e \) does not change on average over one rev.
c.) Calculate change in $a$ in the interval $0 \leq \theta \leq \pi$.

$$\left\langle \frac{da}{dt} \right\rangle_{0,\pi} = \frac{1}{\pi} \int_{0}^{\pi} \frac{2 a^{3/2} a_{L} \sin \theta}{\sqrt{\mu(1-e^{2})}} d\theta$$

$$= \frac{1}{\pi} \left[ \frac{2 a^{3/2} a_{L} (-\cos \theta)}{\sqrt{\mu(1-e^{2})}} \right]_{0}^{\pi}$$

$$= \frac{4}{\pi} \frac{a^{3/2} a_{L}}{\sqrt{\mu(1-e^{2})}}$$

So approx. change in $a$ is

$$\Delta a = \left\langle \frac{da}{dt} \right\rangle_{0,\pi} \times \frac{1}{2} \pi$$

$$= \frac{4}{\pi} \frac{a^{3/2} a_{L}}{\sqrt{\mu(1-e^{2})}} \times \pi \sqrt{\frac{a^{3}}{\mu}}$$

$$\Delta a = \frac{4 a^{3} a_{L}}{\mu \sqrt{1-e^{2}}}$$

4. Suppose that the satellite in Prob. 1 needs to change its inclination by thrusting orbit-normal (i.e., perpendicular to the orbital plane). Calculate the change in inclination $i$ for the following cases ($\omega = \pi/2$):

a.) thrust turned on between $\theta = 0$ and $\theta = \pi$,

b.) thrust turned on between $\theta = \pi/4$ and $\theta = 3\pi/4$.

c.) Why are these inclination changes not very different?

Solu. $a_{e} = a_{r}$ \hspace{0.5cm} (a$_{r}$ = 0, a$_{\theta}$ = 0)

a.) $\left\langle \frac{d i}{dt} \right\rangle_{0,\pi} = \frac{1}{\pi} \int_{0}^{\pi} \left( \frac{\Delta}{dt} \right) d\theta$

$$= \frac{1}{\pi} \int_{0}^{\pi} \left( \frac{a(1-e^{2})}{\mu} \cdot \frac{\cos (\theta + \omega)}{1+e \cos \theta} a_{e} \right) d\theta$$
\[ \langle \frac{d\phi}{dt} \rangle_{0,\pi} = \frac{1}{\pi} \int_{0}^{\pi} \sqrt{\frac{a(1-e^2)}{\mu}} \frac{a_T (-\sin \theta)}{1 + e \cos \theta} \, d\theta \]

Let \( \mu = [1 + e \cos \theta] \),
then \( d\mu = -e \sin \theta \, d\theta \)

\[ \langle \frac{d\phi}{dt} \rangle_{0,\pi} = \frac{1}{\pi} \int_{0}^{\pi} \sqrt{\frac{a(1-e^2)}{\mu}} a_T \frac{1}{e} \ln (1 + e \cos \theta) \bigg|_{0}^{\pi} \]

\[ \langle \frac{d\phi}{dt} \rangle_{0,\pi} = \frac{1}{\pi} \int_{0}^{\pi} \sqrt{\frac{a(1-e^2)}{\mu}} a_T \frac{1}{e} \left[ \ln (1-e) - \ln (1+e) \right] \]

To evaluate this for small \( e \), use Taylor expansion for \( \ln \)

\[ \ln (1+x) = x - \frac{1}{2} x^2 + \cdots \]

\[ \ln (1-x) = -x - \frac{1}{2} x^2 + \cdots \]

So, for small \( e \), \( \ln (1-e) \approx -e \)
\( \ln (1+e) \approx e \)

\[ \Longrightarrow \langle \frac{d\phi}{dt} \rangle_{0,\pi} \approx \frac{1}{\pi} \int_{0}^{\pi} \sqrt{\frac{a(1-e^2)}{\mu}} a_T \frac{1}{e} (-e-e) \]

\[ = -\frac{2}{\pi} \sqrt{\frac{a(1-e^2)}{\mu}} a_T \]

Change in \( i \) is then

\[ \Delta i = \langle \frac{d\phi}{dt} \rangle_{0,\pi} \times \frac{1}{e} T \]

\[ = -\frac{1}{\pi} \sqrt{\frac{a(1-e^2) \pi}{\mu}} a_T \]

\[ \Delta i \approx -\frac{2 a^2}{\mu} a_T \]

To understand why \( i \) changes, need to examine the torque created by the thrust. Torque \( \vec{M} = \vec{r} \times \vec{F} \), where \( \vec{F} = \text{thrust} \)

dividing by \( e \) must give \( \vec{M} = \vec{r} \times \vec{a_T} \)

(remember, central gravitational field exerts NO torque!)
b.) \( \left\langle \frac{d\dot{i}}{dt} \right\rangle \mid \!_{\frac{\pi}{4}, \frac{3\pi}{4}} = \frac{1}{\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\!_{\frac{\pi}{4}}} \int_{\frac{\pi}{4}}^{3\pi/4} \frac{a_i(1-e^2) a_T}{\mu} \sin \Theta \frac{d\Theta}{1+e \cos \Theta} \)

\[ = \frac{2}{\pi} \sqrt{\frac{a_i(1-e^2) a_T}{\mu}} \frac{1}{e} \ln \left( \frac{1+e \cos \Theta}{1-e \cos \Theta} \right) \frac{3\pi/4}{\pi/4} \]

\[ = \frac{2}{\pi} \sqrt{\frac{a_i(1-e^2) a_T}{\mu}} \frac{1}{e} \left[ \ln \left( \frac{1-e \frac{1}{e} \frac{1}{2}}{1+e \frac{1}{e} \frac{1}{2}} \right) - \ln \left( \frac{1+e \frac{1}{e} \frac{1}{2}}{1-e \frac{1}{e} \frac{1}{2}} \right) \right] \]

\[ \approx \frac{2}{\pi} \sqrt{\frac{a_i(1-e^2) a_T}{\mu}} \frac{1}{e} \left[ -e \frac{1}{2} - e \frac{1}{2} \right] \]
The change in \( i \) is then
\[
\Delta i = \frac{2}{\pi} \sqrt{\frac{a(1-e^2)}{\mu}} a_T \left[ -V^2 \right] \times \frac{1}{4} \frac{T}{T} \quad \frac{1}{4} T = \frac{\pi}{2} \sqrt{\frac{a^3}{\mu}}
\]

\[
= \frac{2}{\pi} \sqrt{\frac{a}{\mu}} a_T \pi \frac{1}{2} \frac{a^3}{\mu} \sqrt{2}
\]

\[
\Delta i = -V^2 \frac{a^2}{\mu} a_T
\]

Use \( \frac{1}{4} T \) since thrust is on from
\[
\theta = \frac{\pi}{4} \quad \text{to} \quad \frac{3\pi}{4}
\]
which is \( \frac{1}{4} \text{ rev.} \)

c.) The change in \( i \) is only 30% less than in part b)
because the torque causing the orb. plane to rotate is more effective now \( \theta = \frac{\pi}{2} \).

\[
\overrightarrow{h} \Delta \overrightarrow{h}
\]

\[
\overrightarrow{a_T} \Delta \overrightarrow{a_T}
\]

Only the component of \( \overrightarrow{m} \) that is perpendicular to the node line will cause an inclination change.
That component is \( m \sin \theta \)
Since orb. is approx. circular, \( r \) is const.

\[
\sin \theta
\]

And so, the integrated effect of the torque occurs mostly in the range
\[
\theta = \frac{\pi}{2}
\]
\[
\frac{\pi}{4} \quad \frac{3\pi}{4}
\]

area \( = \sqrt{2} \)

area \( = 2 \)