A Derivation of $B(z) = \frac{\mu_0 i}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} \hat{k}$

We want to find the magnetic field $B(z)$ at the point $Q$ due to the circular current loop carrying a current $i$ in the direction shown in the Figure. The plane of the circle is the $x - y$ plane, and the point $Q$ is at a height $z$ above the center of the circle. The vector, which is from the center of the circle to the point on the circle is $R$; $z$ is the vertical vector (in the positive $z$ direction) which starts at the center of the circle; and the vector which makes an angle with the plane of the circle is $r$. ($r$ slopes upward.) The small vector whose tail is at $Q$ is the magnetic field $B(z)$ at the point $z$.

\[ B(z) = \int d\mathbf{B} = \int \frac{\mu_0 i}{4\pi} \frac{ds \times \mathbf{r}}{(z^2 + R^2)^{3/2}} = \]

\[ = \int \frac{\mu_0 i}{4\pi} \frac{ds \times (z - \mathbf{R})}{(z^2 + R^2)^{3/2}} = \]

\[ = \int \frac{\mu_0 i}{4\pi} \frac{ds \times (-\mathbf{R})}{(z^2 + R^2)^{3/2}}, \quad (1) \]

where $ds$ is the infinitesimal displacement along the circle in the direction of the current, which is located at the tail of $r$. In the last step we have used:

\[ \int ds \times z = (\int ds) \times z = 0. \]

Now

\[ \int ds \times (-\mathbf{R}) = (\int ds \cdot R) \hat{k} = 2\pi R^2 \hat{k}. \]

Substitution of this result into eqn. (1) gives the desired result.