Dynamic cash discounts when sales volume is stochastic

Jeffrey R. Stokes*

The Pennsylvania State University, University Park, PA 16802, USA

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Abstract

In this paper, a dynamic terms of sale model is developed which suggests deep cash discounts can be partially explained by the positive relationship between the shadow value of sales and the optimal cash discount. The effect of sales volume uncertainty on the magnitude of cash discounts is also explored. Numerical results suggest the relationship between uncertainty and cash discounts is nonlinear. The model is then re-cast as a dynamic, differential game between two competing suppliers who use cash discounts to entice buyers. The results suggest that when firms are allowed to behave strategically, cash discounts are always larger as a result.

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1. Introduction

Trade credit represents one of the most flexible sources of short-term financing available to firms principally because it arises spontaneously with the firm’s purchases (Scott, Martin, Petty, & Keown, 1999).1 Estimates from Dunn and Bradstreet and Robert Morris Associates

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1 Trade credit is distinguished from consumer credit in this paper through the extension of credit terms between two firms rather than between a firm and a consumer.

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suggest that the typical firm offering trade credit has an investment in accounts receivable that represents about 25% of all assets. Naturally, the management of accounts receivable becomes increasingly important the more the firm relies on credit sales.

The decision to offer trade credit and the determination of the firm’s terms of sale are important managerial considerations. In addition, the purchasing firm’s decision to take advantage of a cash discount or not and the motivations behind such a decision are also important. Retail firms such as Wal-Mart and Kroger have become particularly adept at exploiting the advantages of trade credit by moving product inventory well before the last day that a discount can be taken and thereby earning considerable return on the float.

Survey research conducted by the Credit Research Foundation found that nearly 60% of respondents offered cash discounts to their customers. Of the customers offered cash discounts, about 43% responded that over 75% of their customers took them. In addition, respondents reported that they felt the level of their cash discount accelerated DSO by as much as 20 days. Survey evidence presented in Progressive Grocer suggests that in 2000, demanding better cash discount terms was ranked the 10th most likely action to be taken by grocers in 2001 while supplier cash discounts were ranked 22nd in terms of problem severity.

While results such as these indicate the importance of trade credit and cash discounts, there are other reasons cash discounts are important such as the Robinson–Patman Act (RPA). The RPA precludes firms from price discrimination including price discrimination that can arise through differential credit terms. However, there is evidence that suggests firms may at times violate the RPA without being caught. Two separate surveys reported in Supermarket Business suggest that as high as 76% of manufacturers thought a stronger enforcement of the RPA would be beneficial (see Partch, 1990 and Partch, 1992). The Federal Trade Commission’s investigation into flavorings and spice marketer McCormick and Co. is a recent example where potential violations of the RPA occurred through preferential cash discounts.

Numerous factors likely influence the determination of the firm’s terms of sale; especially the level of the firm’s cash discount. Building on work by Nadiri, Wrightsman, and Schwartz, Hill and Riener (H&R) model the firm’s optimal cash discount in a static and deterministic setting by assuming that a greater proportion of the firm’s customers will pay early (and hence take a cash discount) the higher the cash discount offered by the firm. The H&R model, while intuitive and probably the most cited work in the cash discount area, typically predicts cash discounts that are lower than those observed in practice.

Interestingly, about two-thirds of the respondents indicated that they do not re-evaluate their cash discount policy as market conditions change.

McCormick signed a settlement agreement in 2000 with the FTC following a four-year investigation.

Hill and Riener’s static cash discount model suggests optimal cash discounts according to the equation: 
\[ \delta^* = \frac{1}{2}[1 - (1 + r/365)^{-m/n}] \] when the cash discount does not affect sales volume. In the equation, \( \delta^* \) represents the optimal cash discount percentage, \( m \) is the last day the discount can be taken, \( n \) is the last day payment in full can be made (\( m < n \)), and \( r \) represents the firm’s annual cost of capital. Hill and Riener also present a model to determine the maximum cash discount a firm should offer when current sales volume is positively impacted by the cash discount offered by the firm. This is the model used by Borde and McCarty. However, the maximum cash discount is likely a much less useful number than the optimal cash discount and the impact a cash discount has on future sales volume is explicitly ignored by H&R due to their static framework.
More recent research in the trade credit area has tended to focus less on modeling the
determination of optimal cash discounts and more on explanations for why firms engage
in financial intermediation by offering trade credit in the first place. Ferris suggests firms
offer trade credit because doing so lowers transactions costs by separating the exchange of
goods from the exchange of money. Emery’s work implies financial market imperfections
motivate some firms to offer trade credit. In both cases, uncertainty plays a role in explaining
the existence of trade credit but there is no indication as to how uncertainty affects trade
credit terms including the cash discount. In addition, there is no indication of the role that
supplier competition plays in explaining the existence of trade credit or the size of cash
discounts. This, despite many textbooks suggesting that supplier competition is important
for the firm’s optimal cash discount (see, for example, Scherr or Maness and Zietlow).

In one of the more recent attempts to explain the existence of trade credit, Smith contends
that firm’s offer credit terms with implicitly high interest rates to screen potential buyers’
default risk. By offering trade credit, Smith contends that firms can identify prospective de-
faults more quickly than if short term financing is provided by financial institutions. Among
other things, Smith’s model suggests that trade credit terms (most notably cash discounts)
will be somewhat uniform within an industry, but can vary considerably, especially across
industries. In addition, Smith’s model appears to be the only trade credit model wherein
any interaction (supplier/purchaser in this case) is modeled explicitly.

Even so, the approaches taken by Smith and others leave many unanswered questions
with regard to why firms offer trade credit and in a more normative sense, how cash discounts
should be determined in practice. Thus far, the trade credit literature has generally ignored
the impact of dynamics, temporal uncertainty, and supplier competition. One notable ex-
ception is the recent work of Rien er who examines the impact of growth and seasonality on
the optimal time the firm should change its credit policy using a deterministic, but dynamic
model. Rien er’s work is especially important because it appears to be one of the only models
of credit policy that fully embrace the dynamic nature of the firm. However, it is likely that
the response of a change in credit policy is stochastic. Further, futures sales volume is also
stochastic and it would appear that it might be possible to glean more information about
credit policy decisions by breaking out of Rien er’s deterministic setting.

Given the importance of trade credit and the fact that some features of the firm’s cash
discount problem have not been adequately addressed in the literature (e.g. sales volume
uncertainty and supplier competition), the purpose of this research is to investigate the
economics of cash discounts. To do this, an analytic model of the firm’s optimal cash discount
is developed in a dynamic and stochastic framework within the paradigm of maximizing
firm net present value. The decision theoretic model that results nests a well-known static
model of optimal cash discounts and suggests a linkage between the firm’s shadow value of
sales and the optimal cash discount. In addition, the model allows for a characterization of
the impact of sales volume and sales volume uncertainty on the firm’s optimal cash discount.
The model is then extended to capture the existence of supplier competition as an oft cited
reason for why firms offer cash discounts and as a relatively simple explanation for why

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5 As is well known, credit terms of 2/10 net 30 are implicitly equivalent to short-term financing at excessively
high effective interest rates. High enough, Smith argues, to screen potential buyers’ default risk.
cash discounts observed in practice are generally higher than those suggested by existing models.

2. Optimal dynamic cash discounts

Hill and Riener correctly hypothesize the existence of costs and benefits of cash discounts. Clearly the cost to a supplier of offering a cash discount is the reality that some percentage of the invoice amount is forgone if customers opt to take the discount. On the benefits side however, cash discounts can accelerate cash inflows thereby reducing the firm’s need to borrow for investment purposes. Because cash discounts essentially amount to an aggregate price reduction across all items sold, additional sales volume is likely stimulated as well. If cash discounts induce buyers to pay earlier, they may also have the effect of reducing potential bad debt losses. In addition to these benefits, firms likely offer cash discounts with implicitly high interest rates to screen buyer default probability (as in Smith), to match competitor discounts, and more simply, to maintain buyer relationships.

In this section a decision theoretic model is presented that is consistent with H&R’s assumptions. Because firms continually face stochastic sales volume, a dynamic model is specified where it is assumed that future sales volume is influenced by the firm’s current choice of cash discount. Given the advantages of the stochastic calculi for incorporating uncertainty into dynamic economic models, a stochastic optimal control framework is appropriate.

Let $t$ represent time and assume a firm determines their cash discount in a manner consistent with the paradigm of wealth maximization. That is, the firm seeks a cash discount that maximizes the present value of their income flow represented by sales revenue less the variable cost of sales. Let $p(\delta)_t$ be the proportion of the firm’s customers that will take a cash discount of $\delta_t$ percent at time $m$ if offered such a discount at time $t$ ($<m$). Also, let $dp/d\delta_t > 0$ so that higher cash discounts induce a greater proportion of the firm’s customers to take the discount. Further, assume that customers not taking the cash discount either pay in full at time $n$ ($n > m$) or do not pay at all. Let $b(\delta_t)$ represent the proportion of customers who result in bad debt losses. Discounts do not likely increase bad debt losses so that $db/d\delta_t \leq 0$ is assumed.

The firm’s instantaneous flow of gross sales revenue, $R(S_t)dt$, is then given by

$$R(S_t)dt = \{p(\delta)_t S_t (1 - \delta_t) e^{-rm} + [1 - p(\delta)_t - b(\delta_t)] S_t e^{-rm}\} dt$$

where $S_t$ is a state variable representing time $t$ sales volume. The first term in (1) represents revenue from customers taking advantage of the discount while the second term represents revenue from those not taking advantage of the discount. We assume bad debts cannot be (or are too expensive to be) collected.

Let $\nu$ represent a constant fractional sales margin, $l$ represent the time at which the firm must pay for the goods they sell, and $r$ represent the firm’s cost of capital. Further, let $C(S_t) = \nu S_t e^{-rl}$ represent the present value of the firm’s cost of goods sold. Assuming the present is time zero, the discounted flow of the firm’s net revenue stream up to any time $T$ is given by

$$\int_0^T [R(S_t) - C(S_t)] e^{-rt} dt. \tag{2}$$
Using the definitions of $R(S_t)$ and $C(S_t)$, substituting in Eq. (2), and recasting the problem in the context of the control variable $\delta_t$ results in the following objective functional

$$V(S_t, t) = \max_{\delta_t \in [0,1]} \mathbb{E}_0 \int_0^T \{ p(\delta_t)S_t(1 - \delta_t) e^{-rt} \}
+ [1 - p(\delta_t) - b(\delta_t)]S_t e^{-rn} - vS_te^{-rl} \} e^{-rt} dt$$

where $V(S_t, t)$ represents the value functional and $\mathbb{E}_0$ is an expectation operator taken over the stochastic sales volume state variable. It is also important to point out that the firm is assumed to not exert any discernable degree of market power over its purchasers and visa versa.

### 2.1. Sales volume dynamics

The firm’s sales volume is assumed to evolve over time according to the following stochastic differential equation

$$\frac{dS_t}{S_t} = g(\delta_t)dt + \sigma(\delta_t)dZ_t,$$

which implies that the expected rate of growth in sales volume is equal to $g(\delta_t)$ and $\sigma(\delta_t)$ is the instantaneous volatility in sales volume. Initially, we allow for the possibility that both the expected growth rate and volatility are dependent on the trade discount policy chosen by the firm. While $dg/d\delta_t > 0$ is plausible, it is not altogether clear what should be the appropriate sign for $d\sigma/d\delta_t$.

It should be noted that with $dg/d\delta_t > 0$, we are assuming that as the firm optimally adjusts their cash discount, there is a sustained impact on the expected rate of growth in sales volume. Technically, the impact on the rate of growth in sales volume is only truly sustained if the firm maintains this cash discount level. Even so, the assumption is entirely consistent with past research including Hill and Rienier who assume there is a sustained increase in sales when a firm increases their cash discount.

An alternative to the previous assumption is the case where a change in the firm’s cash discount level has only a one-time change in the firm’s sales volume level. While there is no empirical evidence to suggest that either case (i.e. one-time versus permanent change) prevails, a priori, a relatively strong case can be made for the permanent change since the cash discount is merely a reduction in price which should stimulate demand. However, to model the alternative, one might augment the sales volume state equation to $dS_t = \kappa dQ_t$ where $dQ_t$ is Poisson and equals one with probability $\lambda(\delta_t)dt$ and equals zero with probability $1 - \lambda(\delta_t)dt$ with $d\lambda/d\delta_t > 0$. In this way, higher cash discounts raise the probability...

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6 Wrightsman suggested static models to determine (non-simultaneously) the determination of an optimal cash discount and optimal credit length period ($m$ using the present notation). The present model could be specified to determine the optimal credit length period simultaneously with the optimal cash discount but would abstract from some of the more intuitive results that follow given such a first order condition cannot be solved explicitly for $m$.

7 With $dg/d\delta_t > 0$, the higher the discount offered by the firm, the higher the expected rate of growth in sales volume. This assumption is intuitive since a higher cash discount is equivalent to a reduced price which should stimulate demand.
of boosting sales volume via a jump of amplitude $\kappa$ while lower cash discounts have an opposing effect. More importantly, changing the firm’s cash discount has no guaranteed impact on the firm’s sales volume and if sales volume jumps, it still continues to grow at the same rate. Most likely, a model of this sort will suggest that the magnitude of the optimal cash discount is positively related to the probability of realizing a jump in sales volume. While such a model is an interesting approach to the optimal cash discount problem, it is beyond the scope of the present model.

2.2. Decision theoretic cash discounts

Given the preceding, the firm’s decision theoretic trade credit problem can be specified as the following relatively simple stochastic optimal control problem

$$V(S_t, t) = \max_{\delta_t \in [0, 1]} E_0 \int_0^T \left\{ p(\delta_t) S_t (1 - \delta_t) e^{-\gamma t} + \left[ 1 - p(\delta_t) - b(\delta_t) \right] S_t e^{-\gamma t} - v S_t e^{-\gamma t} \right\} e^{-\gamma t} dt$$

subject to:

$$\frac{dS_t}{S_t} = g(\delta_t) dt + \sigma(\delta_t) dZ_t.$$  

We assume $V(S_t, t)$ is increasing and concave in $S_t$ (i.e. $\partial V/\partial S > 0, \partial^2 V/\partial S^2 < 0$) as required for a bounded solution. It is worth noting that the system appearing in Eqs. (5) and (6) significantly extends H&H by casting the cash discount problem in a dynamic and stochastic framework (H&R is static and deterministic), and operationalizing the conventional wisdom with regard to the linkage between sales volume dynamics and the firm’s optimal choice if cash discount.

Assumptions for how the firm’s cash discount affects $p(\delta_t), b(\delta_t), g(\delta_t),$ and $\sigma(\delta_t)$ are required to generate specific information about the solution of the system. To facilitate an analytic investigation of the system, we assume $p(\delta_t) = \pi \delta_t, b(\delta_t) = 0, g(\delta_t) = \gamma \delta_t,$ and $\sigma(\delta_t) = \sigma.$ While these are relatively simple specifications, they are likely not unrealistic. The value of $p(\delta_t)$ must lie in the $[0, 1]$ interval and is necessarily equal to zero when $\delta_t = 0$ and equal to one when $\delta_t = 1.$ Because the effect of cash discounts on bad debts is not a focus of the model, it is assumed $b(\delta_t) = 0.$ The assumptions regarding $g(\delta_t)$ and $\sigma(\delta_t)$ imply that the expected rate of sales volume growth is impacted by the discount policy but that the volatility is not.

Making these substitutions where appropriate, the Hamilton–Jacobi–Bellman (HJB) equation is

$$-V_t = \max_{\delta_t \in [0, 1]} \left\{ [\pi \delta_t S_t (1 - \delta_t) e^{-\gamma t} + (1 - \pi \delta_t) S_t e^{-\gamma t} - v S_t e^{-\gamma t}] e^{-\gamma t} + \gamma \delta_t S_t V_S + \frac{1}{2} (\sigma S_t)^2 V_{SS} \right\}$$

8 It is likely the case that some $\delta_t < 1$ induces all customers who plan on paying to pay early implying that $\pi > 1.$ Hill and Riner assume this functional form with $\pi = 1$ while Rashid and Mitra assume $\pi = 10$ using the same functional form in static models of optimal cash discounts.

9 This assumption is also consistent with Hill and Riner.
where \(-V_t = -\partial V/\partial t, V_S = \partial V/\partial S\) and \(V_{SS} = \partial^2 V/\partial S^2\). As the assumption of a finite planning horizon does not necessarily result in a more realistic model and likely induces a more difficult control problem, it is assumed the firm has an infinite planning horizon so that the value functional can be written as \(V(S_t, t) = J(S_t)e^{-rt}\). Eq. (7) then becomes

\[
rJ = \max_{\delta_t \in [0,1]} \left[ \pi \delta_t S_t(1 - \delta_t)e^{-rm} + (1 - \pi \delta_t)S_t e^{-m} - vS_t e^{-rt} \right. \\
+ \left. \gamma \delta_t S_t J_S + \frac{1}{2}(\sigma S_t)^2 J_{SS} \right].
\] (8)

The interpretation of the value functional \(J(S)\) is that it is the value of the firm. That is, the present value of the firm’s sales volume stream when an optimal cash discount policy is followed.

Differentiating the right hand side of Eq. (8) with respect to the control and solving the resulting equation for \(\delta_t\) gives

\[
\delta^*(S_t) = \frac{1}{2} [1 - e^{-r(n-m)}] + \left(\frac{\gamma e^m}{2\pi}\right) J_S.
\] (9)

As shown, the optimal discount policy is a linear function of the firm’s shadow value of sales with slope equal to \(\gamma e^m/2\pi\) and intercept equal to \(1/2[1 - e^{-r(n-m)}]\). The intercept term in Eq. (9) is identical to H&R’s static solution (with the exception of continuous discounting—see Eq. (11) on page 70 of H&R’s article) for the case when a cash discount affects the timing of purchaser payments but not the volume of sales. The slope term arises in the dynamic case because the cash discount chosen by the firm at time \(t\) impacts future sales volume through the diffusion equation (6) when \(\gamma > 0\). As shown, the larger the shadow value of sales (\(J_S\)), the higher the discount the firm is willing to offer.

Notice also in Eq. (9) that if the expected rate of growth in sales volume is unrelated to the firm’s cash discount [i.e. if \(g(\delta_t) = \gamma\)], the second term in (9) disappears and the H&R static solution still holds. The primary implication of this result is that stochastic sales volume and a dynamic modeling approach are themselves insufficient to undermine the validity of H&R static solution.\(^{11}\) In the present model, a direct relationship between the firm’s expected rate of growth in sales volume and cash discount policy is required for a deviation from the H&R solution. In a later section of the paper, supplier competition is introduced as another potential reason for a cash discount policy that deviates from H&R.

2.2.1. Certainty

To solve the system, Eq. (9) is substituted into Eq. (8) and the resulting second order, nonlinear, ordinary differential equation (ODE) solved to recover \(J(S_t)\). After recovering \(J(S_t), J_S\) can be determined as needed by Eq. (9). However, the ODE resulting from substituting Eq. (9) into Eq. (8) does not appear to possess a structure conducive to an analytic solution. This is because the objective functional (5) is quadratic in both state and control

\(^{10}\) The assumption of an infinite planning horizon implies \(T = \infty\) in the upper limit of integration in Eqs. (2), (3) and (5). With \(V(S_t, t) = J(S_t)e^{-rt}\), for some unknown functional \(J(S_t)\), it follows that \(-V_t = re^{-rt}J, V_S = e^{-rt}J_S,\) and \(V_{SS} = e^{-rt}J_{SS}\). Substituting these partial derivatives into Eq. (7) results in Eq. (8).

\(^{11}\) We thank an anonymous reviewer for pointing out this result.
variables. Assuming for the moment that there is no uncertainty results in the deterministic solution \( J(S_t) = \xi S_t \) where \( \xi \) is a constant that is the solution to the quadratic equation:

\[
\hat{a} \xi^2 + \hat{b} \xi + \hat{c} = 0,
\]

with

\[
\hat{a} = \left[ \frac{\gamma^2 e^{rm}}{2} \right], \quad \hat{b} = \gamma \delta_{HR} - r, \quad \hat{c} = \pi (1 - \delta_{HR}) \delta_{HR} e^{rm} + (1 - \pi \delta_{HR}) e^{rm} - ve^{-rt},
\]

representing the H&R static solution.

Therefore, in a deterministic setting, the optimal dynamic cash discount policy (9) can be written as

\[
\delta^*(S_t) = \delta^* = \delta_{HR} + \left( \frac{\gamma e^{rm}}{2\pi} \right) \xi,
\]

a constant. This particularly simple result suggests that the larger the firm’s shadow value of sales, the higher the cash discount the firm should offer. Optimal dynamic cash discounts are essentially a grossed up H&R cash discount.

This result offers more economic intuition for observed differential cash discounts within an industry beyond firms just having different costs of capital and payment deadlines. A firm with a relatively large shadow value of sales will experience a large increase in firm value for each incremental product sale. Such may be the case in relatively competitive industries characterized by newer firms that have not realized their full potential in terms of generating sales volume. These firms follow an aggressive cash discount policy to attract new business and stimulate sales volume.

Also notice that only in the case when the firm’s shadow value of sales is zero does the model imply that the static and dynamic cash discounts are equal. Older, more established firms with a level of sales volume reflective of slower growth would likely have lower shadow value of sales and therefore feel less pressure to pursue an aggressive cash discount policy. Rather, these firms would opt for the H&R cash discount or something close to it.

2.2.2. Uncertainty

While intuitive, the preceding results suggest that the firm’s optimal cash discount is unrelated to the firm’s level of sales volume and is constant over time. In fact, the linkage between sales volume and the firm’s cash discount has not been established in the literature with existing models. Ceteris paribus, higher sales volume can be indicative of increasing market share, market power, as well as industry maturity. The preceding results suggest that a firm should offer a higher cash discount (and thereby stimulate additional future sales volume) the higher their shadow value of sales. It remains to show the role that uncertainty plays in linking the firm’s sales volume with its optimal cash discount.

As it turns out, the fact the firm faces some uncertainty with regard to the effect of their cash discount policy on future sales volume is enough to insure that the value functional is concave in sales volume. To see this, Eq. (9) can be substituted into Eq. (8) and the resulting
ODE solved numerically. Doing so results in the graph depicted in Fig. 1.\textsuperscript{12} As shown in Fig. 1, when $\sigma > 0$, $J(S)$ is convex in $S$ for low sales volumes and concave in $S$ for higher sales volumes. That is, when $J_{SS} > 0$ ($J_{SS} < 0$), optimal cash discounts are increasing (decreasing) in sales volume. Also, the firm offers the highest cash discount when $J_{SS} = 0$. Also shown in Fig. 1 is the static H&R solution which corresponds to the case where $J_S = 0$. Therefore, only firm’s with a shadow price of sales equal to zero should set their cash discount equal to what the static H&R suggests.

Shown in Fig. 2 is the effect of volatility on the optimal cash discount policy for different sales volume levels. The impact is nonlinear suggesting that, in general, low (high) sales volume volatility implies increasing (decreasing) cash discounts. Therefore, one response suggested by the model is that when firms face uncertainty with regard to the impact their cash discount will have on future sales volume, they tend to set higher (lower) cash discounts when current sales volume volatility is relatively low (high). Further, the results suggest that the more sales volume the firm is generating, the less likely the firm is to adjust their cash discount policy in response to volatility.

\textsuperscript{12} Substituting Eq. (9) into Eq. (8) results in a second order, nonlinear, ordinary differential equation of the form: $J_{SS} = f(J, J_S, S)$. The solution functional $J(S)$ can be approximated by determining coefficients, $a_i$, of the polynomial $J(S) = \sum_{i=0}^{I} a_i S^i$ for suitably large $I$. Given the specification for $J(S)$, the functional relation above can be re-written as $\left(\sum_{i=0}^{I} a_i S^i\right)'' = f \left[\sum_{i=0}^{I} a_i S^i, \left(\sum_{i=0}^{I} a_i S^i\right)', S\right] + \varepsilon(S)$ where primes denote differentiation and $\varepsilon(S)$ is an error term that insures the equality holds. This co-location method of approximating a solution to an ODE is implemented by minimizing the sum of squared errors for each sales volume level considered and results in Fig. 1.

\textsuperscript{13} It should also be noted that each $S$ appearing in Fig. 1 is from the same–time period. That is, Fig. 1 shows the relationship between $S$ and $J(S)$ at a single point in time, not over time. However, given the expected rate of growth in sales volume is positive in Eq. (4), the time path of the value functional and optimal cash discount policy are likely similar to those presented in Fig. 1.
Fig. 2. (a) Relationship between optimal cash discount, sales volume volatility, and sales volume (View 1). (b) Relationship between optimal cash discount, sales volume volatility, and sales volume (View 2).

The suggestion by Weston and Brigham, as well as Smith, that more risk and uncertainty imply deep cash discounts is consistent with the results presented here, but generally only when sales volume uncertainty is relatively low to begin with and sales volume itself is low as well. The model suggests the opposite for excessively high volatility environments, namely, that the firm should consider lowering its cash discount when faced with additional uncertainty.

As an alternative view of the relationship between uncertainty and cash discounts, it may be the case that when firm’s increase cash discounts substantially, they are actually signaling an increase in sales volume uncertainty from a relatively low level or perhaps a decrease in future sales volume. In the former case, a firm in an industry characterized by very low sales volume volatility may over-react to a perceived increase in volatility by increasing
their cash discount level substantially. In the latter case, a firm facing a precipitous drop is sales volume may react by increasing their cash discount.

2.2.3. Sales volume and cash discount dynamics

It is also interesting to point out that sales volume dynamics conditional on an optimal policy are given by

$$\frac{dS_t}{S_t} = \gamma \delta H + \left( \gamma \pi r \right) J S \, dt + \sigma S_t \, dZ_t.$$  \hspace{1cm} (13)

As shown in Eq. (13), the higher the firm’s shadow value of sales, the higher the anticipated rate of growth in sales volume. Only when the shadow value of sales is zero does the firm have an expected rate of growth in sales volume that is equal to that which would prevail if the firm followed the H&R optimal cash discount policy.

The dynamics of the optimal cash discount policy can also be determined by applying Ito’s lemma to Eq. (9) and substituting where necessary using Eq. (13). The result is

$$d\delta_t = \left( \gamma \pi r \right) \left[ \gamma \delta_t S_t J S S + \left( \sigma S_t^2 \right) J S S S \right] dt + \left( \gamma \pi r \right) \sigma S_t J S S dZ_t.$$  \hspace{1cm} (14)

This equation shows that the firm’s optimal cash discount policy can be described by a stochastic differential equation that depends on the second and third derivatives of the value functional as well as all the other parameters in the model. Given the results of Fig. 1, $J S S S$ can be either positive or negative given the sign of the drift term in Eq. (14) changes as sales volume changes. However, the equation does highlight the complexity of the dynamics of the firm’s optimal cash discount policy.

2.3. Game theoretic cash discounts

In the certain, infinite horizon case, the decision theoretic model presented above suggests that firm’s should adopt a constant cash discount policy that is analogous to the static discount policy plus an amount that is related to the firm’s shadow value of sales volume. In the uncertain case, the firm’s optimal cash discount policy is convex in low sales volume and concave in high sales volume. In general, the more sales volume uncertainty the firm faces, the higher (lower) the cash discount the firm should offer for relatively low (high) volatility environments.

While the decision theoretic model suggests cash discounts that change continually as sales volume changes, this is potentially inconsistent with evidence published by the Credit Research Foundation who reported that over 80% of the respondents indicated that they do not reevaluate their cash discounts as interest rates change. Perhaps more importantly, 64% indicated that they do not reevaluate their cash discount as market conditions change. The reluctance on the part of the firm to change cash discounts is likely attributable to many things including profit margins, risk, buyer relations, and perhaps most importantly, supplier competition.

The decision theoretic model presented predicts a wide range of behavior depending on the firm’s cost of capital, sales margin, and actual sales volume, but does not ex-
plicitly suggest how supply firms engaged in competition react to one another and what the impact such reaction has on cash discounts. Supply firms such as those being modeled compete with one another for the purchasing firm’s dollar. Also, observed cash discounts can vary within an industry and are generally higher than most decision theoretic models suggest. As shown in Fig. 1, as sales volume increases, cash discounts fall to less than 2% and fall to near zero quickly thereafter. Therefore, a legitimate question to ask is why a firm with relatively high sales volume would offer a substantial cash discount?

One possibility for the existence of terms such as 2/10 net 30 is that firms compete with one another when offering cash discounts and this results in an industry practice. In this context, two firms, say firm $i$ and firm $j$, both sell a relatively homogeneous product and seek to improve their respective sales volumes by setting their discounts strategically. In this section, the decision theoretic cash discount model of the previous section is recast to allow for strategic behavior. The nature of the behavior is that firm $i$, when trying to decide on what discount to offer, takes into account firm $j$’s most likely or observed cash discount and sets their own cash discount accordingly. However, firm $j$ plays the game analogously and what results is a differential game where each player behaves strategically at each instant in time. The resulting Nash equilibrium provides useful insight into the role that supplier competition and strategy play in setting cash discounts.

The system comprised of Eq. (5) subject to Eq. (6) is assumed to be the problem faced by both firms. However, in the strategic case, the proportion of customers taking the discount is assumed to be dependent on the discount policy of both firms. Let $p(\delta_i, \delta_j)$ represent the proportion of firm $i$’s customers taking an early discount with $\partial p/\partial \delta_i > 0$ and $\partial p/\partial \delta_j < 0$. The sign of the first partial derivative indicates that, as before, a greater proportion of firm $i$’s customers will take the cash discount the greater the discount. The second partial derivative indicates that if firm $j$ raises their discount (relative to firm $i$’s), there is a negative impact on the proportion of customers taking firm $i$’s discount that is attributable to customers (costlessly) switching from firm $i$ to firm $j$.

A simple example should clarify this point. Suppose firm $i$ has 100 customers and a discount policy such that 50% of their customers take the discount and 50% do not take the discount. If firm $j$ raises their discount relative to firm $i$, some of firm $i$’s customers, say 10, who would have taken the discount offered by firm $i$, will now buy product from firm $j$ and take firm $j$’s discount. Assuming that none of the 50 customers who were not taking firm $i$’s discount switch to firm $j$, firm $i$ is left with $40/90 = 44.4\%$ taking the discount and $50/90 = 55.6\%$ not taking the discount. It is in this manner that the proportion of firm $i$’s customers switch in response to firm $j$’s policy.

A second change in the system is that the expected rate of growth in sales is also assumed to be impacted by both firm’s discount. Recall, the expected rate of growth was $g(\delta_i)$ in the decision theoretic case and we now specify $g(\delta_i, \delta_j)$ with $\partial g/\partial \delta_i > 0$ and $\partial g/\partial \delta_j < 0$. Therefore, when firm $i$ raises its discount relative to that of firm $j$, firm $i$’s expected rate of growth in sales increases. However, when firm $j$ increases its discount relative to that of firm $i$, firm $i$’s expected rate of growth in sales is diminished.

The impact of these changes is in the specification of the functional characterizing the proportion of customers taking the cash discount in Eq. (5) and the drift term in Eq. (6). Maintaining the other assumptions, let the proportion of the firm’s customers
taking the discount be: \( p(\delta_i^t, \delta_j^t) = \pi \delta_i^t + \mu (\delta_i^t - \delta_j^t) \) where \( \pi, \mu > 0 \). As shown, the proportion is increasing in the firm’s own cash discount and decreasing in the competing firms cash discount.\(^{14}\) Similarly, let \( dS_i^t = \gamma (\delta_i^t - \delta_j^t) S_i^t dt + \sigma_j^i S_i^t dZ_i^t \) be the sales volume dynamics equation. Notice the expected rate of growth in firm \( i \)’s sales volume is proportional to the difference between the discounts offered by the two firms. If the firms set identical discounts, firm \( i \)’s sales volume is not expected to grow but does evolve stochastically.

In the differential game case, the value function for firm \( i \) must depend on firm \( i \)’s sales volume as well as firm \( j \)’s sales volume (i.e. \( J_i = J_i(S_i, S_j) \)). Making these changes, the HJB Eq. (8) for firm \( i \) is now

\[
rf^i = \max_{\delta_j \in [0, 1]} \left\{ \left[ \pi \delta_i^t + \mu (\delta_i^t - \delta_j^t) \right] \left( 1 - \delta_j^t \right) S_i^t e^{-rt} + \gamma (\delta_i^t - \delta_j^t) S_i^t \right\},
\]

which depends on \( \delta_j^t \), firm \( j \)’s fixed discount policy.\(^{15}\) Differentiating the right hand side of (15) with respect to \( \delta_j^t \), letting \( k = \pi + \mu \), and solving for the optimal cash discount results in

\[
\delta_i^t = \delta_{HR} + \left( \frac{\gamma e^{rm}}{2\lambda S_i^t} \right) (S_j^t J_j^t - S_i^t J_j^t) + \left( \frac{\mu}{2\lambda} \right) \delta_j^t,
\]

where all terms are as previously defined.\(^{16}\)

Eq. (16) can be rearranged slightly for a more intuitive interpretation. Letting \( \partial J_i / \partial S_i (S_i^t / J_i) = \eta_{S_i}^i \) represent the firm’s own sales elasticity and \( \partial J_i / \partial S_j (S_j^t / J_i) = \eta_{S_j}^i \) represent the firm’s cross sales elasticity, Eq. (16) can be expressed as

\[
\delta_i^t = \delta_{HR} + \left( \frac{\gamma e^{rm}}{2\lambda} \right) \left( \eta_{S_i}^i - \eta_{S_j}^i \right) + \left( \frac{\mu}{2\lambda} \right) \delta_j^t.
\]

As shown, firm \( i \)’s strategic cash discount policy is dependent on three distinct components. The first component is purely non-reactiary and is the H&R cash discount, \( \delta_{HR} \). This component forms the base cash discount from a decision theoretic standpoint. The remaining two components have reactionary components and always add to this base cash discount.

\(^{14}\) The specification of a proportion that depends directly on both the firm’s discount policy and the difference between the firm’s discount policy and it’s competitor’s insures that a positive proportion of the firm’s customers take the firm’s cash discount even when firm \( i \) and firm \( j \) have identical discounts.

\(^{15}\) To arrive at Eq. (15), it was assumed that firm \( j \)’s sales volume dynamics evolve in a manner analogous to that of firm \( i \) [i.e. like Eq. (6)] and that the sales volumes for the two firms are correlated. The notation \( J_{S_i S_j} \) means the cross partial derivative of firm \( i \)’s value functional with respect to firm \( j \)’s sales volume.

\(^{16}\) To solve this system, both firms’ optimal policies can be substituted into Eq. (15) and the equation analogous to Eq. (15) for firm \( j \). The result is a system of two, second order, nonlinear, partial differential equations. The solution is not taken up here, as there is enough intuition that can be gleaned from the analytic representations of the optimal policies.
They arise because sales volume is positively related to the cash discount (as before) but also because of the assumption that firm \( i \) reacts strategically to firm \( j \) when setting their cash discount.

The second term in Eq. (17) is partially reactionary and partially non-reactionary. The term is always non-negative because firm \( i \) has a non-decreasing shadow value of own sales and a non-increasing shadow value of competitor sales. As before, the greater the firm’s shadow value of sales, the greater the optimal cash discount. However, in the strategic case there is an added effect attributable to competitor sales volume. The more sensitive firm \( i \)’s value functional to its competitor’s sales volume, the higher the cash discount firm \( i \) will offer. Such may be the case for highly competitive industries characterized by close substitutes. If firm \( i \) knows that the present value of their net income flow is greatly impacted by an increase in firm \( j \)’s sale volume, firm \( i \) will offer a higher cash discount than under less competitive conditions.

The remaining component in Eq. (17) is a purely reactionary term representing the direct relationship between the two firms’ cash discount policies. Notice when \( \mu = 0, \lambda = \pi \) and Eq. (17) reduces to Eq. (9) as there is no reason for firm \( i \)’s value functional to depend on firm \( j \)’s sales volume in such a case. However, when \( \mu > 0 \) firm \( j \)’s discount policy matters to firm \( i \) when firm \( i \) sets their cash discount. The extent of the importance is measured by the coefficient appearing on the \( \bar{\delta}^j \) term, namely, \( \mu/2\lambda \). The larger is \( \mu \), the more important firm \( j \)’s cash discount is to firm \( i \). The limit of the coefficient is \( 1/2 \) implying that in the most competitive situation possible, firm \( i \) will set its discount policy approximately equal to their H&R decision theoretic value, add an amount to account for the own and cross sales elasticities, and then add one-half of firm \( j \)’s discount to this amount.\(^{17}\)

Firm \( j \), however, is also faced with the same cash discount problem and therefore has a HJB equation like Eq. (15) and optimal cash discount policy analogous to Eq. (16). This implies that firm \( j \) will react to firm \( i \)’s choice of cash discount and will adjust their cash discount accordingly.

As an example of the type of behavior supported by the model, suppose firm \( i \)’s H&R decision theoretic policy and the own and cross sales elasticities suggest that a one-percent discount is appropriate and the firm elects to adopt a 1/10 net 30 policy. Further, assume that firm \( j \) is an otherwise identical firm who observes firm \( i \)’s policy and sets their 10 net 30 policy cash discount in response at 1% plus 0.5% (i.e. 1% plus half of firm \( i \)’s discount). Firm \( i \), upon observing (or logically concluding) that firm \( j \) is offering 1 1/2/10 net 30, will respond by offering 1 3/4/10 net 30, etc., until the strategic equilibrium policy of 2/10 net 30 results. The strategic behavior of the two firms insures that higher cash discounts result than the decision theoretic model suggests as long as the coefficient \( \mu/2\lambda \) is positive. That is, as long as a competitor’s cash discount policy matters to the firm.

Also important to note is that unless some sort of structural change occurs, neither firm has an incentive to deviate from the strategic equilibrium by changing their cash discount. Using simulation, Borde and McCarty suggest the Hill and Riener solution is very stable (insensitive to input data) and use this as motivation for why we do not observe firm’s

\(^{17}\) This result is found by taking the limit of the coefficient on the \( \bar{\delta}^j \) term in Eq. (17). It should also be noted that while \( \lambda \), and hence \( \mu \) appears in the denominator of the third term in Eq. (17), the term does not necessarily vanish as \( \mu \to \infty \) because it also appears in the functional \( J^i_0 \).
changing their cash discounts frequently. The present model suggests an alternative reason for such behavior. Namely, that upon reaching a strategic equilibrium, there is little incentive for either firm to change their policy.

While this symmetric result is appealing as an intuitive mechanism for understanding the type of behavior suggested by the model, the fact remains that no two firms are likely “otherwise identical”. The differential game above can support the observation that differential cash discounts are possible even in a strategic equilibrium because firms differ in their ability to access capital and among other things, can have different profit margins (implying $v^i \neq v^j$), and sales volumes. While this likely has little if any effect on the form of the value functional facing each firm, it does imply that the first (non-reactionary) and second (partially reactionary) terms in Eq. (17) for each firm can be quite different. Consequently, two firms can offer different cash discount policies even in the strategic equilibrium.

As an example, suppose that firm $i$ and firm $j$ have the same access to capital and that their H&R cash discounts are equal but that firm $j$ is an industry leader with relatively high sales volume. Firm $i$ has significantly lower sales volume and high own sales and cross sales elasticities. In this case, firm $i$ will opt for a higher cash discount than firm $j$. For example, firm $i$ may have a base cash discount of 2.5% to firm $j$’s 1.0%. In strategic equilibrium (with $\mu/2\lambda$ equal to 1/2), firm $i$ will offer a 4% discount (2.5% + 1.5%) while firm $j$ will optimally offer a 3% discount (1.0% + 2.0%) in response.

Also interesting to note is the implication of firm $i$’s cash discount being positively related to that of firm $j$ in the event that firm $j$ lowers their discount. Conventional wisdom might suggest that firm $i$ would not change their discount and thereby increase their future sales volume and presumably their market share as a result. However, the model suggests an alternative reaction, namely, that firm $i$ would likely also reduce their cash discount in an attempt to establish a new strategic equilibrium.

It is important to note that some sort of structural change that affects the non-reactionary part of firm $j$’s optimal cash discount is necessary to motivate such a move by firm $j$. To see this, assume, as above, that firm $i$ and firm $j$ both have non-reactionary optimal cash discounts of 1%. If the firms are otherwise identical, the strategic equilibrium as presented above is for both firms to offer a 2% cash discount. However, suppose that firm $i$ experiences some sort of structural change that implies that their non-reactionary cash discount should be lowered to 0.5% while firm $j$’s remains at 1%. This reduction could be attributable to any number of things such as a change in the cost of capital, profit margin, an increase in sales volume, or a decrease in sales volume volatility. Firm $j$, upon observing firm $i$’s reduction, will seek to reduce their cash discount in response whereupon firm $i$ will counter with another reduction, etc.

Assuming the limit of the coefficient on the reactionary term in Eq. (17) is one-half, a new strategic equilibrium will obtain wherein firm $i$ will offer a $1\frac{1}{2}$% cash discount and firm $j$ will offer a $1\frac{3}{4}$% cash discount. Therefore the model suggests a relatively counterintuitive response to a firm cutting their cash discount, namely, for a competing firm to also lower theirs. Interestingly, this was precisely the action taken by R.J. Heinz Company in a recent strategy overhaul. Analysts have suggested that Heinz’s decision to cut discounts was in response to similar moves by industry giant Proctor and Gamble (Murphy).
3. Summary and conclusions

Early normative work in the area of trade credit focused on the determination of the firm’s optimal cash discount. Later, more positive work focused on trying to explain why firms engage in financial intermediation by offering trade credit in the first place. However, it is unclear whether any of the existing models establish an analytic linkage between sales volume and cash discounts or uncertainty and cash discounts. In addition, typical normative models suggest optimal cash discounts that can be significantly less than those observed in practice.

The research presented in this paper fills a void in the trade credit literature by suggesting that the firm’s optimal cash discount policy is fundamentally related to the firm’s shadow value of sales volume. A firm with a high shadow value of sales volume will offer higher cash discounts than a firm with low shadow value of sales volume. Further, the relationship between sales volume volatility and the optimal cash discount is nonlinear with low sales volume volatility positively related to the cash discount and high sales volume volatility negatively related to the cash discount.

Even so, the decision theoretic model presented may fall short of truly explaining why firms offer such high cash discounts in practice. Recasting the model in the context of a differential game, while adding significantly to the complexity of the model, supports the intuitive idea that when firms compete with cash discounts, they will generally offer higher cash discounts than in the purely decision theoretic case. The optimal policy suggests that firms should offer a discount approximate to their decision theoretic value and then adds some fraction (the limit of which is 1/2) of their competitor’s discount. The strategic Nash equilibrium also suggests differential, yet stable cash discounts are possible, which are both consistent with observation. Lastly, the model suggests that if some structural change occurs and one firm must cut its discount, a competitor firm should also cut their discount in an effort to establish a new strategic equilibrium. Such counter intuition has not been established in previous research.

References


