To demonstrate the semi-implicit numerical method, we consider a simplified model for 1D, single-phase flow in a pipe. The differential equations for this model are listed below.

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho V = 0, \quad (2 - 86)
\]

\[
\frac{\partial \rho e}{\partial t} + \nabla \cdot \rho e V = -p \nabla \cdot V + h(T_w - T), \quad (2 - 87)
\]

and

\[
\frac{\partial V}{\partial t} + V \nabla V = -\frac{1}{\rho} \nabla p - KV |V|. \quad (2 - 88)
\]

Here, \( K \) is a wall friction coefficient that may be a function of velocity and fluid properties, \( h \) is a heat-transfer coefficient multiplied by the heat-transfer area per volume of fluid, and \( T_w \) is a pipe wall temperature.

A staggered spatial mesh is used for the finite-volume equations, with thermodynamic properties evaluated at the cell centers and the velocity evaluated at the cell edges. Only difference equations on the 1D version of this mesh are demonstrated, but the generalization to 2D and 3D versions is not difficult. To ensure stability, flux terms at cell edges use donor-cell averages of the form

\[
\langle YV \rangle_{j+\frac{1}{2}} = \begin{cases} Y_j V_{j+\frac{1}{2}} & , \quad V_{j+\frac{1}{2}} \geq 0 \\ Y_{j+1} V_{j+\frac{1}{2}} & , \quad V_{j+\frac{1}{2}} < 0 \end{cases}, \quad (2 - 89)
\]

Here \( Y \) may be any state variable. With this notation the 1D finite-difference divergence operator is

\[
\nabla_j \cdot (YV) = \frac{(A_{j+\frac{1}{2}} \langle YV \rangle_{j+\frac{1}{2}} - A_{j-\frac{1}{2}} \langle YV \rangle_{j-\frac{1}{2}})}{\text{vol}_j}, \quad (2 - 90)
\]

where \( A \) is the area of the cell edge and \( \text{vol}_j \) the cell volume. The term \( V \nabla V \) becomes

\[
V_{j+\frac{1}{2}} \Delta x_{j+\frac{1}{2}} \nabla_j \cdot (YV) = \begin{cases} V_{j+\frac{1}{2}} \frac{V_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} & , \quad V_{j+\frac{1}{2}} \geq 0 \\ V_{j+\frac{1}{2}} \frac{V_{j+\frac{1}{2}} - V_{j-\frac{1}{2}}}{\Delta x_{j+\frac{1}{2}}} & , \quad V_{j+\frac{1}{2}} < 0 \end{cases}, \quad (2 - 91)
\]

where \( \Delta x_{j+\frac{1}{2}} = 0.5 \ (\Delta x_j + \Delta x_{j+1}) \). This choice of \( \Delta x_{j+\frac{1}{2}} \) for Eq. (2 - 91) is necessary for
more accurate calculation of pressure drops in pipes modeled with a nonuniform mesh
than is provided with a donor cell $\Delta x_{j+1/2}$.

Semi-Implicit Difference Equations

\[
\left(\frac{V_{j+1/2}^{n+1} - V_{j+1/2}^{n}}{\Delta t}\right) + V_{j+1/2}^{n} \nabla_{j+1/2} \tilde{V}^{n+1} \\
+ \beta \left(\frac{V_{j+1/2}^{n+1} - V_{j+1/2}^{n}}{\nabla_{j+1/2}^{n+1} \tilde{V}^{n}}\right) \\
+ \frac{1}{(p)_{j+1/2}^{n} \nabla x_{j+1/2}^{n+1}} (\tilde{P}_{j+1}^{n+1} - \tilde{P}_{j}^{n+1}) \\
+ K_{j+1/2}^{n} (2V_{j+1/2}^{n+1} - V_{j+1/2}^{n}) V_{j+1/2}^{n} = 0 \quad , \quad (2 - 93)
\]

where

\[
\beta = 0, \quad \nabla_{j+1/2}^{n+1} \tilde{V}^{n} < 0 \\
1, \quad \nabla_{j+1/2}^{n+1} \tilde{V}^{n} > 0
\]

\[
\left(\frac{\rho_{j+1/2}^{n+1} - \rho_{j+1/2}^{n}}{\Delta t}\right) + \nabla_{j+1/2} \left(\rho^{n} V^{n+1}\right) = 0 \quad ; \quad (2 - 94)
\]

\[
\left(\frac{\tilde{\rho}_{j+1/2}^{n+1} \epsilon_{j+1/2}^{n+1} - \rho_{j+1/2}^{n} \epsilon_{j+1/2}^{n}}{\Delta t}\right) + \nabla_{j+1/2} \left(\rho^{n} \epsilon^{n} V^{n+1}\right) \\
+ \tilde{\rho}_{j+1/2}^{n+1} \nabla_{j+1/2} (V^{n+1}) - \tilde{\epsilon}_{j+1/2}^{n} (T_{wj}^{n} - \tilde{T}_{j+1/2}^{n+1}) = 0 \quad ; \quad (2 - 95)
\]

Ignore tilde’s above a variables. They are part of notation in a more complex two-step method.

The first step in the solution of the semi-implicit equation set rearranges the motion equation to obtain the new-time velocity as a linear function of new time pressures. For Eq. (2 - 93), this step results in the relation

\[
V_{i}^{n+1} = \frac{V_{i}^{n} - \Delta t \left[ V_{i}^{n} \nabla_{i} (\tilde{V}^{n+1} - \beta \tilde{V}^{n}) - K_{i}^{n} V_{i}^{n} V_{i}^{n} \right] + \frac{\tilde{P}_{i+1}^{n+1} - \tilde{P}_{i}^{n+1}}{\langle \rho \rangle_{i}^{n} \Delta x_{i}}}{1 + \Delta t (2K_{i}^{n} V_{i}^{n} + \beta V_{i} \tilde{V}^{n})} \quad , \quad (2 - 98)
\]
where $i = j + 1/2$. Given this relation, the derivatives of velocity with respect to pressure are

$$
\frac{dV_{j+1/2}^{n+1}}{dP_j^{n+1}} = \frac{\Delta t}{\rho_i^n \Delta x_i \left(1 + 2 K_i^n \Delta t \left| V_i^n \right| + \Delta t \beta V_i \bar{V}_i^n \right)} \quad (2 - 99)
$$

and

$$
\frac{dV_{j+1/2}^{n+1}}{dP_j^{n+1}} = -\frac{dV_{j+1/2}^{n+1}}{dP_j^{n+1}} . \quad (2 - 100)
$$

Equation (2 - 98) and thermodynamic equations giving $\rho(P,T)$ and $e(P,T)$ are substituted into Eqs. (2 - 94) and (2 - 95) to give a coupled system of nonlinear equations with unknowns $P_{j}^{n+1}$ and $T_{j}^{n+1}$. Solution of this system is obtained with a standard Newton iteration. Given the latest estimates $P_{j}^{n+1}$ and $T_{j}^{n+1}$ for pressures and temperatures, we assume the solution is

$$
P_{j}^{n+1} = P_{j}^{n+1} + \delta P_{j} \quad (2 - 101)
$$

$$
T_{j}^{n+1} = T_{j}^{n+1} + \delta T_{j} . \quad (2 - 102)
$$

After we substitute Eqs. (2 - 98) through (2 - 102) into the basic equation set, making the necessary Taylor series expansions, and discard nonlinear terms in $\delta P_{j}$ and $\delta T_{j}$, we find that the resulting linearized mass and energy equations are:

$$
\frac{\partial \rho^{'}}{\partial T} \bigg|_{j}^{n+1} \delta T_{j} + \frac{\partial \rho^{'}}{\partial P} \bigg|_{j}^{n+1} \delta P_{j} - \left( \frac{A_{j+1/2}^{n+1} \rho_{j+1/2}^{n} \Delta t}{\text{vol}_j} \frac{dV_{j+1/2}^{n+1}}{dP_{j+1/2}^{n+1}} \right) (\delta P_{j+1} - \delta P_{j})
$$

$$
- \left( \frac{A_{j-1/2}^{n} \rho_{j-1/2}^{n} \Delta t}{\text{vol}_j} \frac{dV_{j-1/2}^{n+1}}{dP_{j-1/2}^{n+1}} \right) (\delta P_{j} - \delta P_{j-1})
$$

$$
= \rho_{j}^{n} - \rho_{j}^{n+1} - \Delta t \nabla \cdot (\rho_{j}^{n} V_{j}^{n+1}) \quad (2 - 103)
$$

and

$$
\left( \rho_{j}^{n+1} \frac{\partial e^{'}}{\partial T} \bigg|_{j}^{n+1} + e_{j}^{n+1} \frac{\partial \rho^{'}}{\partial T} \bigg|_{j}^{n+1} + \Delta t \bar{h}_{j}^{n+1} \right) \delta T_{j}
$$

$$
+ \left( \rho_{j}^{n+1} \frac{\partial e^{'}}{\partial P} \bigg|_{j}^{n+1} + e_{j}^{n+1} \frac{\partial \rho^{'}}{\partial P} \bigg|_{j}^{n+1} + \Delta t \bar{V}_{j} \cdot V_{j}^{n+1} \right) \delta P_{j}
$$

$$
- \left( P_{j}^{n+1} + P_{j+1/2}^{n} e_{j+1/2}^{n} \right) \left( \frac{A_{j+1/2}^{n+1} \Delta t}{\text{vol}_j} \frac{dV_{j+1/2}^{n+1}}{dP_{j+1/2}^{n+1}} \right) (\delta P_{j+1} - \delta P_{j}) \quad (2 - 104)
$$
\[
- \left( P_{j}^{n+1} + \rho_{j}^{n} e_{j-1/2}^{n} \right) \left( A_{j-1/2} \Delta t \frac{dV_{j-1/2}^{n+1}}{dP_{j}^{n+1}} \right) \left( \delta P_{j} - \delta P_{j-1} \right) \\
= \rho_{j}^{n} e_{j}^{n} - \rho_{j}^{n+1} e_{j}^{n+1} - \Delta t \left[ \nabla_{j} : \rho_{j}^{n} e_{j}^{n} V^{n+1} \right] + P_{j}^{n+1} \nabla_{j} \cdot V^{n+1} - \tilde{h}_{j}^{n} \left( T_{w,j} - T_{j}^{n+1} \right).
\]

The normal procedure for starting this linearization is to make an initial estimate for the new-time pressure and temperature of \( P^{n+1} = \tilde{P}^{n} \) and \( T^{n+1} = \tilde{T}^{n} \). When SETS is used, however, an extra call to the thermodynamic subroutines can be saved by taking \( P^{n+1} = \tilde{P}^{n} \) and \( T^{n+1} = \tilde{T}^{n} \).

Equations (2 - 103) and (2 - 104) can be abbreviated as

\[
\mathbf{B}_{j} \begin{pmatrix} \delta P_{j} \\ \delta T_{j} \end{pmatrix} = b + c_{j} \left( \delta P_{j+1} - \delta P_{j} \right) - d_{j} \left( \delta P_{j} - \delta P_{j-1} \right), \quad (2 - 105)
\]

where \( \mathbf{B} \) is a \( 2 \times 2 \) matrix. For the 1D numerics, this equation is multiplied by \( B^{-1} \) (accomplished with a linear system solver), which yields

\[
-d_{1,j}^{\prime} \delta P_{j-1}^{\prime} + (1 + d_{1,j}^{\prime} + c_{1,j}^{\prime}) \delta P_{j} - c_{1,j}^{\prime} \delta P_{j+1} = b_{1,j}^{\prime}, \quad (2 - 106)
\]

and

\[
\delta T_{j} = b_{2,j}^{\prime} + c_{2,j}^{\prime} (\delta P_{j+1} - \delta P_{j}) - d_{2,j}^{\prime} (\delta P_{j} - \delta P_{j-1}), \quad (2 - 107)
\]

where \( b_{j}^{\prime} = B^{-1} b \), etc. All coefficients are stored, and then the set of equations represented by Eq. (2 - 106) is solved for all \( \delta P_{j} \). Finally, the known values for pressure variations are substituted into Eq. (2 - 107) to obtain temperature changes and used with Eqs. (2 - 99) and (2 - 100) to obtain updated velocities. Given these changes, the next estimates for new-time pressures and temperatures are generated and used to obtain densities and energies. If changes in \( \delta T \) and \( \delta P \) are too large, these estimates are used to return to Eqs. (2 - 101) and (2 - 102) to relinearize for another iteration of the 1D equations.