Von Neumann Stability Analysis

Fourier Series

For Periodic Boundary Conditions over length L, a function \( f(x,t) \) can be expressed as the series:

\[
f(x,t) = \sum_{n=0}^{\infty} a_n(t) \ e^{\frac{i 2\pi nx}{L}} \tag{1}
\]

Define \( k_n = \frac{2\pi n}{L} \) as the wave number

A complex exponential can be related to trigonometric function by the equation

\[ e^{i \theta} = \cos \theta + i \sin \theta. \]

You can prove this equality with a Taylor expansion of all terms.

Linearize a convection equation with constant velocity:

\[
\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + V \frac{\rho_j^n - \rho_{j+1}^n}{\Delta x} = 0
\]

with the substitution

\[
\rho \rightarrow \rho_o + \delta \rho
\]

and assume that the set of \( \rho_o \)'s satisfy the original difference equation. Then the linearized difference equation is:

\[
\frac{\delta \rho_j^{n+1} - \delta \rho_j^n}{\Delta t} + V \frac{\delta \rho_j^n - \delta \rho_{j+1}^n}{\Delta x} = 0
\]

By doing this we are trying to discover whether small deviations from the correct solution return to the solution, or grow.

Expand the perturbations as a Fourier Series and consider a specific wave number contribution

\[
\delta \rho_j^n \rightarrow \delta \rho \ e^{i k (j-1)\Delta x}
\]

the equation becomes

\[
\frac{\delta \rho_j^{n+1} e^{i k (j-1)\Delta x} - \delta \rho_j^n e^{i k (j-1)\Delta x}}{\Delta t} + \frac{V}{\Delta x} (\delta \rho_j^n e^{i k (j-1)\Delta x} - \delta \rho_j^n e^{i k (j-2)\Delta x}) = 0
\]
Factoring yields

\[ \delta \rho^{n+1} - \delta \rho^n \left[ 1 - \frac{V \Delta t}{\Delta x} (1 - e^{-ik \Delta x}) \right] - 0 \]

\[ \delta \rho^{n+1} = \left[ 1 - \frac{V \Delta t}{\Delta x} (1 - e^{-ik \Delta x}) \right] \delta \rho^n \]

The Von Neumann stability criteria states that the difference equations are stable if when put in the form

\[ \delta \rho^{n+1} - G \delta \rho^n \]

\[ |G| \leq 1 \]

For this scalar form of G this condition is met if

\[ GG^* \leq 1 \]

let \((1 - e^{-ik \Delta x}) = \beta\)

\[ \beta^* = 1 - e^{ik \Delta x} \]

\[ \beta + \beta^* = 2 - 2 \cos (k \Delta x) \]

\[ \beta \beta^* = 2 - 2 \cos (k \Delta x) \]

Therefore, for this case

\[ GG^* = 1 + \left[ \left( \frac{V \Delta t}{\Delta x} \right)^2 - \frac{V \Delta t}{\Delta x} \right] [2 - 2 \cos(k \Delta x)] \]

and \(GG^* \leq 1\)

when \[ \left[ \left( \frac{V \Delta t}{\Delta x} \right)^2 - \frac{V \Delta t}{\Delta x} \right] [2 - 2 \cos(k \Delta x)] \leq 0 \]

or \[ \left[ \frac{V \Delta t}{\Delta x} - 1 \right] \leq 0 \]

or \[ \frac{V \Delta t}{\Delta x} \leq 1 \]

When G is a matrix rather than a scalar quantity the stability condition can be shown to be true if the magnitude of all eigenvalues of G are all less than or equal to zero.