Advection-Diffusion Equation

\[ \frac{d}{dt} \rho + V \frac{d}{dx} \rho = k \frac{d^2}{dx^2} \rho \]

Look at a general form of an Explicit difference equation

\[ \frac{\rho(x, t + \Delta t) - \rho(x, t)}{\Delta t} + V \frac{w \rho(x + \Delta x, t) + (1 - w) \rho(x, t) - w \rho(x, t) - (1 - w) \rho(x - \Delta x, t)}{\Delta x} = k \left( \frac{\rho(x + \Delta x, t) + 2 \rho(x, t) + \rho(x - \Delta x, t)}{\Delta x^2} \right) \]

Substitute in a perturbed density

\[ \rho(x, t) = \rho_0 + \delta \rho(t) e^{ikx} \]

and divide the entire equation by \( e^{ikx} \)

\[ \frac{\delta \rho(t + \Delta t) - \delta \rho(t)}{\Delta t} + \delta \rho(t) \frac{w e^{-ik\Delta x} + 1 - 2w - (1 - w) e^{ik\Delta x}}{\Delta x} = k \delta \rho(t) \left( \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2} \right) \]

let \( c = \frac{V \Delta t}{\Delta x}, \gamma = \frac{k \Delta t}{\Delta x^2}, \) and \( \Theta = k \Delta x \)

\[ \delta \rho(t + \Delta t) = \left[ 1 + 2\gamma \left( \cos(\Theta) - 1 \right) - c(2w - 1) \left( \cos(\Theta) - 1 \right) \right] \delta \rho(t) \]

The amplification factor is

\[ G = \left[ 1 + \left( 2\gamma - 2w - c \right) \left( \cos(\Theta) - 1 \right) \right] - c \sin(\Theta) \]

Examining G, stability is not possible if the magnitude of either the real or imaginary parts exceeds 1

The imaginary term leads to the requirement that \( c \) be less than one

\[ \frac{V \Delta t}{\Delta x} \leq 1 \quad \text{solve, } \Delta t \leq \frac{\Delta x}{V} \]

For the real term to be less than one we need:

\[ 2\gamma - 2w - c \geq 0 \]

which restricts possible weighting factors. For the real term to be greater than -1 we need:

\[ 1 - 2 \left( 2\gamma - 2w - c \right) \geq -1 \]

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or

\[ \Delta t \leq \frac{\Delta x^2}{2k - 2w \cdot V \cdot \Delta x + V \cdot \Delta x} \]
Note that the previous two conditions are necessary for stability but not sufficient. Fine tuning the stability conditions requires a little more finesse.

The product of $G$ and its complex conjugate is

$$\left[1 + (2 \gamma - 2 \cdot w \cdot c + c) \cdot \cos(\Theta) - 1\right]^2 + c^2 \left[1 - \cos^2(\Theta)\right]$$

Let $\mu = \cos(\Theta)$ and look the extreme values of $|G|^2$ with respect to $\mu$

$$\frac{d}{d\mu} \left[\left[1 + (2 \gamma - 2 \cdot w \cdot c + c) \cdot (\mu - 1)\right]^2 + c^2 \left(1 - \mu^2\right)\right] = 0$$

Solve for $\mu$ with respect to

$$\frac{1}{4} \left(\frac{-2 \cdot \gamma + 2 \cdot w \cdot c - c}{\gamma^2 - 2 \cdot \gamma \cdot w \cdot c + \gamma \cdot c + w^2 \cdot c^2 - w \cdot c^2}\right)$$

Try the case of central differencing

$$\frac{1}{4} \left(\frac{-2 \cdot \gamma + 2 \cdot w \cdot c - c}{\gamma^2 - 2 \cdot \gamma \cdot w \cdot c + \gamma \cdot c + w^2 \cdot c^2 - w \cdot c^2}\right)$$

This gives a extreme for $|G|^2$ when

$$\mu = \frac{4 \cdot \gamma^2 - 2 \gamma}{4 \cdot \gamma^2 - c^2} \quad \text{or} \quad \mu = \frac{4 \cdot \gamma^2 - 2 \gamma}{4 \cdot \gamma^2 - 2 \cdot \gamma + (2 \cdot \gamma - c^2)}$$

or

$$\mu = \frac{1}{1 + \frac{2 \cdot \gamma - c^2}{4 \cdot \gamma^2 - 2 \cdot \gamma}}$$

To make useful statements about stability we need a general idea of the behavior of $|G|^2$. First notice that $|G|^2 \geq 0$ for $-1 \leq \mu \leq 1$, and when $|G|^2 = 1$

$$\frac{d}{d\mu} \left[\left[1 + 2 \cdot \gamma \cdot (\mu - 1)\right]^2 + c^2 \left(1 - \mu^2\right)\right]$$

Simplify, collect $\mu$ and substitute $\mu = 1$

$$\left[1 + 2 \cdot \gamma \cdot (\mu - 1)\right] + c^2 \left(1 - \mu^2\right)$$

With bounding values and the slope at one end of the interval, the next thing to explore is the location and nature of the maximum or minimum of $|G|^2$. First note that $4 \gamma^2 - 2 \gamma$ is always less than or equal to zero since $\gamma \leq \frac{1}{2}$

When $2 \cdot \gamma - c^2 \geq 0$ or $c^2 \leq 2 \gamma$ $\mu$ of the extreme value is greater than one. It is a maximum
rather than minimum because the derivative of $|G|^2$ is positive and $|G|^2 = 1$ when $\mu = 1$. Further behavior this situation can be studied in three regions.

Region 1

\[ 0 \geq \frac{2 \gamma - c^2}{4 \gamma^2 - 2 \gamma} \geq -1 \quad \text{corresponding to} \quad c^2 \leq 2 \gamma \quad \text{and} \quad c \leq 2 \gamma \]

In this case $|G|^2$ is monotonically increasing between $\mu = -1$ and $\mu = 1$, so has a maximum value of 1 (Remember that $|G|^2$ is greater than zero between $\mu = -1$ and $\mu = 1$).

Region 2

\[ -1 > \frac{2 \gamma - c^2}{4 \gamma^2 - 2 \gamma} > -2 \]

Here $|G|^2$ has a minimum at $\mu < -1$. This means that the maximum value of $|G|^2$ must again be 1 at $\mu = 1$.

Region 3

\[ -2 \geq \frac{2 \gamma - c^2}{4 \gamma^2 - 2 \gamma} > -\infty \]

Here a local minimum exists in $|G|^2$ at a value of $\mu$ between -1 and 0. Since $|G|^2$ is always greater than zero, the largest possible values of $|G|^2$ are at $\mu = -1$ or $\mu = 1$. At $\mu = -1$

\[
|G|^2 = \left[ 1 + 2 \gamma \cdot (-2) \right]^2 \leq 1 \quad \text{which requires} \quad \gamma \leq \frac{1}{2} \quad \text{or} \quad \Delta t \leq \frac{\Delta x^2}{2k}
\]

What happens when $c^2 > 2 \gamma$ Look at the details of the derivative of $|G|^2$

\[
\frac{d}{d\mu} \left[ \left[ 1 + 2 \gamma (\mu - 1) \right]^2 + c^2 (1 - \mu^2) \right] \quad \text{simplify collect } \mu \rightarrow (8 \gamma^2 - 2 c^2) \cdot \mu + 4 \gamma - 8 \gamma^2 = 0
\]

When $\mu = 1$ and $c^2 > 2 \gamma$ this slope is negative. However, G is one when $\mu$ is one, so G must be greater than one for some range of $\mu$.

All of this leads us to two conditions that are necessary and sufficient for stability. The first is the standard condition for conduction:

\[ \Delta t \leq \frac{\Delta x^2}{2k} \]

and from $c^2 \leq 2 \gamma$ we get:

\[ \Delta t \leq \frac{2 \cdot k}{V^2} \]

Note that for a pure convection problem ($k=0$) this gives the result that central differencing is unstable.

Also note that this is not the full story. The cell Peclet number (or cell Reynolds number is defined as:

\[ Pe = \frac{2 \cdot c}{\gamma} \quad \text{or} \quad Pe = \frac{V \cdot \Delta x}{k} \]

we will later see that boundary conditions can add a stability requirement $Pe \leq 2$.