Chapter 7

Measuring Power spectrum and Bispectrum

In this chapter, we provide a practical guide for measuring power spectrum and bispectrum by using Fast Fourier Transforms. First, we show the method of calculating the power spectrum and bispectrum from N-body simulations, where input density contrast field is defined in a well defined cubic box. Then, we move to the power spectrum estimation for general cases and show the way we implement the FKP estimator (Feldman et al., 1994) with Fast Fourier Transform. For those two cases, we make special emphases on the normalization coefficients, the shot noise correction and the window function.

7.1 Power spectrum and Bispectrum from N-body simulation

Estimating power spectrum and bispectrum from the N-body simulation data is less complicated as N-body simulations have 1) the cubic box, 2) the constant mean number density. We divide the general procedure of measuring power spectrum and bispectrum from N-body simulation by following five steps:

(1) Distributing particles onto the regular grid
(2) Fourier transformation
(3) Estimating power spectrum and bispectrum
(4) Deconvolving window function
(5) Subtracting shot noise

7.1.1 From particle to grid

In order to apply the Fast Fourier Transform technique, we have to assign the density field onto each point in the regular grid. The way we distribute a particle to the nearby grid points is called a ‘particle distribution scheme.’ For a given distribution scheme, we
can define an associated ‘shape function’, which quantifies how a quantity (mass, number, luminosity, etc) of particle is distributed. After this process, the sampling we made from the particle distribution is not a mere sampling of the underlying density field, but a sampling convolved with the ‘window function’ of particle distribution scheme. In this section, we shall review the three distribution schemes which are widely used in practice: Nearest-Grid-Point, Cloud-In-Cloud, Triangular-Shape-Cloud. Although we use the particle number density as a representative example below, one can use the same equation for calculating mass weighted power spectrum, luminosity weighted power spectrum, etc, by multiplying the apropos quantity (mass, luminosity) of each particle.

Let us consider the case with $N_p$ particles (e.g. dark matters, halos, galaxies) in a N-body simulation box. The particle number density is given by

$$n_0(x) = \sum_{i=1}^{N_p} \delta^D(x - x_i), \quad (7.1)$$

where $x_i$ is the position of $i$-th particle. Then, the particle number assignment can be formulated by the convolution as

$$n(x) = \int_V d^3x' n_0(x') W(x - x'), \quad (7.2)$$

where $W(x)$ is the window function which quantifies how much of this particle number density is distributed to a grid point separated by $x$. We sample the continuous number density $n(x)$ by the regular grids of size $N^3$:

$$n^s(x_p) \equiv n(x_p) = \int_V d^3x' n(x') W(x_p - x'). \quad (7.3)$$

Therefore, the sampled density contrast, defined as $\delta^s(x) \equiv n^s(x)/\bar{n} - 1$ is given by the convolution of the real density contrast and the window function,

$$\delta^m(x) = [\delta \star W](x), \quad (7.4)$$

and Fourier transformation of the sampled density contrast is

$$\delta^m(k) = \delta(k) W(k). \quad (7.5)$$

This procedure of convolving with window function can be think of as following. First, we define the cloud shape function (or point spreading function) $S(x')$ of a particle. The shape function can be uniquely determined for given distribution scheme such that the
fraction of the particle number of the particle at \( x \) assigned to the grid point \( x_p \) is given by integrating the shape function within the cubic cell surrounding \( x_p \) (Hockney & Eastwood, 1988, p. 142). That is, the number of a particle at \( x \) assigned to grid point \( x_p \) \( n(x \to x_p) \) is given by

\[
n(x \to x_p) = \int_{|x' - x_p| < H/2} d^3x' S(x' - x).
\]  

(7.6)

In one-dimensional case, the window function can be written in terms of the cloud shape function as

\[
W(x - x_p) = W_p(x) = \frac{1}{H} \int_{x_p - H/2}^{x_p + H/2} S(x' - x)dx',
\]  

(7.7)

where \( H = L/N_{grid} \) is the separation of grids. Using the top-hat function, \( T(x) \),

\[
T(x) = \begin{cases} 
1 & \text{if } |x| < H/2 \\
1/2 & \text{if } |x| = H/2 \\
0 & \text{otherwise}
\end{cases}
\]  

(7.8)

equation (7.7) can be also written as a convolution of top-hat function and cloud function as:

\[
W(x) = \frac{1}{H} \int T\left(\frac{x'}{H}\right) S(x' - x)dx'
\]  

(7.9)

There are three most widely used distribution (window) functions.

7.1.1.1 NGP

Nearest Grid Point (NGP) scheme assigns particles to their nearest grid points. Therefore, the number density changes discontinuously when particles cross cell boundaries. The one dimensional window function for NGP is proportional to the top-hat function

\[
W_{NGP}(x) \equiv \frac{1}{H}T\left(\frac{x}{H}\right) = \begin{cases} 
1/H & \text{if } |x| < H/2 \\
1/(2H) & \text{if } |x| = H/2 \\
0 & \text{otherwise}
\end{cases},
\]  

(7.10)

and its point spreading function is the Dirac delta function as

\[
\frac{1}{H}T\left(\frac{x}{H}\right) = \frac{1}{H}T\left(\frac{x'}{H}\right) \otimes \delta^D(x) = \frac{1}{H}T\left(\frac{x}{H}\right) \otimes \frac{1}{H}\delta^D\left(\frac{x}{H}\right).
\]  

(7.11)

The Fourier Transformation of the top-hat function is the sinc function.

\[
T(k) = \frac{\sin (k/2)}{k/2} = \text{sinc} \left(\frac{k}{2}\right)
\]  

(7.12)
Proof.

\[ T(k) = \int_{-1/2}^{1/2} e^{-ikx} dx = \frac{e^{-ik/2} - e^{ik/2}}{-ik} = \frac{\sin(k/2)}{k/2} \] \hspace{1cm} (7.13)

Therefore, the Fourier Transformation of the NGP window function is

\[ W_{NGP}(k) = \text{FT}[T](Hk) = \text{sinc} \left( \frac{Hk}{2} \right) = \text{sinc} \left( \frac{\pi k}{2k_N} \right) \]

where, \( k_N = \pi/H \) is the Nyquist frequency. I use the similarity theorem of the Fourier transformation.

### 7.1.1.2 CIC

Cloud In Cell (CIC) assignment is the first order distribution scheme which uniformly distributes the particle with top-hat spreading function. In other words, the cloud shape function is given by

\[ S_{CIC}(x) = \frac{1}{H} T \left( \frac{x}{H} \right) . \] \hspace{1cm} (7.14)

Therefore, the window function is

\[ W_{CIC}(x) = \frac{1}{H} T \left( \frac{x}{H} \right) * \frac{1}{H} T \left( \frac{x}{H} \right) , \] \hspace{1cm} (7.15)

and its Fourier Transformation can be simply obtained by the convolution theorem:

\[ W_{CIC}(k) = W_{NGP}(k)^2 = \text{sinc}^2 \left( \frac{\pi k}{2k_N} \right) . \] \hspace{1cm} (7.16)

The explicit expression of the CIC window function is

\[ W_{CIC}(x) = \frac{1}{H} \left\{ \begin{array}{ll} 1 - |x|/H & \text{if } |x| < H \\ 0 & \text{otherwise} \end{array} \right. . \] \hspace{1cm} (7.17)

### 7.1.1.3 TSC

Triangular Shaped Cloud (TSC) scheme is the second order distribution scheme. Its point spreading function is triangular, as the convolution of the two first order (CIC) functions:

\[ S_{TSC}(x) = \frac{1}{H} T \left( \frac{x}{H} \right) * \frac{1}{H} T \left( \frac{x}{H} \right) . \] \hspace{1cm} (7.18)
Therefore, the window function is
\[ W_{TSC}(x) = \frac{1}{H} \mathcal{J}\left(\frac{x}{H}\right) \otimes \frac{1}{H} \mathcal{J}\left(\frac{x}{H}\right) \otimes \frac{1}{H} \mathcal{J}\left(\frac{x}{H}\right), \]  
and its Fourier Transformation is given by
\[ W_{TSC}(k) = W_{NGP}(k)^3 = \text{sinc}^3\left(\frac{\pi k}{2k_N}\right). \]  
The explicit expression of TSC scheme is
\[ W_{TSC}(x) = \frac{1}{H} \begin{cases} \frac{4}{3} - \left(\frac{x}{H}\right)^2, & \text{if } |x| \leq \frac{H}{2} \\ \frac{1}{4} \left(\frac{x}{H} - 1\right)^2, & \text{if } \frac{H}{2} \leq |x| \leq \frac{3H}{2} \\ 0, & \text{otherwise} \end{cases}. \]  

### 7.1.1.4 3D window function

As we use the regular cubic grid, the three dimensional window function is simply given as the multiplication of three one dimensional window functions.
\[ W(x) = W(x_1)W(x_2)W(x_3) \]  
Therefore, its Fourier transformation is
\[ W(k) = \left[ \text{sinc}\left(\frac{\pi k_1}{2k_N}\right) \text{sinc}\left(\frac{\pi k_2}{2k_N}\right) \text{sinc}\left(\frac{\pi k_3}{2k_N}\right) \right]^p, \]  
where \( p = 1, 2, 3 \) for NGP, CIC and TSC, respectively.

### 7.1.2 Power spectrum and bispectrum: the estimators

The power spectrum and bispectrum are defined as
\[ \langle \delta(k_1)\delta(k_2) \rangle = (2\pi)^3 P(k_1)\delta^D(k_1 + k_2) \]  
\[ \langle \delta(k_1)\delta(k_2)\delta(k_3) \rangle = (2\pi)^3 B(k_1, k_2, k_3)\delta^D(k_1 + k_2 + k_3). \]  
Note that both power spectrum and bispectrum are real because of the parity invariance and the reality of the configuration space \( n \)-point correlation function. (See, appendix A of Smith et al. (2008b))

With this definition, \( \sigma_8 \) is
\[ \sigma_8^2 = \int \frac{d^3k}{(2\pi)^3} P(k)|W(kR)|^2 = \int \frac{dk}{k} \Delta^2(k)|W(kR)|^2 \]  
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with $R = 8 \text{Mpc}/h$, and $\Delta^2(k) = P(k)k^3/2\pi^2$ is the dimensionless power spectrum. One can check the normalization of the power spectrum by calculating $\sigma_8$. In some literature, the definition of power spectrum and bispectrum does not contain $(2\pi)^3$, and we have to drop $(2\pi)^3$ in the integration measure of $\sigma_8^2$ equation above.

For the normalization of bispectrum, one can check the value of the reduced bispectrum $Q(k_1, k_2, k_3)$ which is defined as

$$Q(k_1, k_2, k_3) = \frac{B(k_1, k_2, k_3)}{P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1)}.$$

On large scales where so called ‘tree-level’ bispectrum model works, the reduced bispectrum is $4/7$ for the equilateral configuration.

In this section, we shall find the proper normalization to the power spectrum and bispectrum estimators which use the unnormalized Fast Fourier Transformation (FFT) such as FFTW. For denote the unnormalized discrete Fourier transform result by superscript ‘FFTW’.

First, we nondimensionalize the Dirac delta function in the definition of power spectrum and bispectrum. Using the property of delta function,

$$\delta^D(f(x)) = \sum_{x_i \in \text{zeros}} \frac{1}{|f'(x_i)|} \delta^D(x - x_i),$$

we can express the delta function in Fourier space in terms of the Kronecker delta of integer triplet $n_k$. We denote the Kronecker delta for such integer triplet as $\delta^K(n_k)$.

$$\delta^D(k_p) = \delta^D(k_F n_k) = \prod_i \frac{1}{k_F} \delta^K(n_k) = \prod_i \frac{1}{k_F} \delta^K(n_k)$$

7.1.2.1 Power spectrum estimator: direct sampling

FFTW output is the unnormalized DFT, which is related to the sampled Fourier space density field as

$$\delta^{\text{FFTW}}(n_k) = \sum_n \delta(n_p)e^{-i2\pi n_p n_k / N} = \sum_n \delta(n_p) e^{-iK_p r_p} = \frac{\delta(k_p)}{H^3}. \quad (7.26)$$

From the definition of power spectrum

$$\langle \delta(k_1) \delta(k_2) \rangle = H^6 \langle \delta^{\text{FFTW}}(n_1) \delta^{\text{FFTW}}(n_2) \rangle = (2\pi)^3 P(k_1) \delta^D(k_1 + k_2)$$

$$= \frac{(2\pi)^3}{k_F^3} \delta^K(n_1 + n_2) P(k_F n_1), \quad (7.27)$$

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we find the normalization for the power spectrum from DFT as
\[ P(k_{Fn_1}) = \frac{H^6 k_F^3}{(2\pi)^3} |\delta FFFT W(n_1)\delta FFFT W(n_1)| \]  
\[ \left( -n_1 \right), \]  
(7.28) 
where \( V \) is the volume of survey, \( N \) is number of one-dimensional grid, \( H^3 = V/N^3 \) and \( k_F^3 = (2\pi)^3/V \). Therefore, the final estimator for power spectrum is
\[ P(k_{Fn_1}) = \frac{V}{N^6} \left( \delta FFFT W(n_1) \right)^2 = \frac{V}{N^6} \left( \frac{1}{N_k} \sum_{|n_k-n_1| \leq \hat{\delta}} |\delta FFFT W(n_k)|^2 \right), \]  
(7.29) 
where we sum over all Fourier modes within \( k_1 - k_F/2 < |k| < k_1 + k_F/2 \) to estimate the power spectrum at \( k = k_1 = k_{Fn_1} \).

### 7.1.2.2 Bispectrum estimator: direct sampling

From the definition of the bispectrum, we also find
\[ \langle \delta(k_1)\delta(k_2)\delta(k_3) \rangle = H^9 \langle \delta FFFT W(n_1)\delta FFFT W(n_2)\delta FFFT W(n_3) \rangle \]
\[ \approx (2\pi)^3 B(k_1, k_2, k_3)|D(k_{123})| = \left( \frac{2\pi}{k_F} \right)^3 |D(n_{123})|B(k_1, k_2, k_3). \]  
(7.30)

Therefore, the estimator of bispectrum from DFT is
\[ B(k_{Fn_1}, k_{Fn_2}, k_{Fn_3}) \]
\[ = \frac{H^9 k_F^3}{(2\pi)^3} |\delta FFFT W(n_1)\delta FFFT W(n_2)\delta FFFT W(n_3)|D(n_{123}) \]
\[ = \frac{V^2}{N^9} \langle \delta FFFT W(n_1)\delta FFFT W(n_2)\delta FFFT W(n_3) \rangle |D(n_{123})| \]
\[ = \frac{V^2}{N^9} \left( \frac{1}{N_{\text{triangle}}} \sum_{m \in \text{Tri}_{123}} \delta FFFT W(m_1)\delta FFFT W(m_2)\delta FFFT W(m_3) |D(n_{123})| \right), \]  
(7.31) 
where \( \text{Tri}_{123} \) is the set of \( \{ m_1, m_2, m_3 \} \) whose magnitude satisfies \( |m_i - n_i| \leq 1/2 \) and three vectors form a triangle, i.e. \( m_1 + m_2 + m_3 = 0 \).

### 7.1.2.3 Estimating power spectrum II

There is another way of estimating the power spectrum. We slightly change the previous estimator as
\[ P(k_{Fn_1}) = \frac{V}{N^6} \langle \delta FFFT W(n_1)\delta FFFT W(n_2) \rangle |\delta R(n_1 + n_2) \rangle \]
\[ = \frac{V}{N^6} \sum_{m_1 \approx n_1} \sum_{m_2 \approx n_2} \delta FFFT W(m_1)\delta FFFT W(m_2) |\delta R(m_1 + m_2) | \]  
(7.32)
Proof. One can break the summation into two parts, one with positive $m_x$ index, and another with negative $m_x$ index.

$$
\delta_{m_x}(n_r) = \sum_{m \geq n_1} \delta^{FFT}(m) e^{2 \pi i m/N} \\
= \sum_{m \geq n_1} \left[ \delta^{FFT}(m) e^{2 \pi i m/N} + \delta^{FFT}(-m) e^{-2 \pi i m/N} \right] \\
= \sum_{m \geq n_1} \left[ \delta^{FFT}(m) e^{2 \pi i m/N} + \delta^{FFT}(m)^* e^{-2 \pi i m/N} \right] \\
= \sum_{m \geq n_1} \left[ \delta^{FFT}(m) e^{2 \pi i m/N} + c.c. \right] \in \mathbb{R}
$$

In third line, we used the reality of $\delta(r)$, then $\delta(-k) = \delta^*(k)$.

Where, $m_i \simeq n_i$ strictly means that $|m_i - n_i| < s/2$, with $\delta k = sk_F$. Note that

$$
\sum_{m_1 \geq n_1} \sum_{m_2 \geq n_1} \delta^K(m_1 + m_2) = \frac{1}{V_F} \int d^3q_1 \int d^3q_2 \delta^D(q_1 + q_2) \simeq \frac{4 \pi k_F^2 \delta k}{k_F^2} = 4 \pi s n_i^2
$$

is the total number of independent $k$-modes inside the spherical shell of radius $k$ and width $\delta k = sk_F$. We shall use the total number of $k$ modes here instead of the actual number of $k$ modes, because when we do the inverse DFT later, both $\delta(k)$ and $\delta(-k)$ shall be summed over.

Using the orthonormality of DFT

$$
\delta^K(n_p + n_q) = \frac{1}{N^3} \sum_{n_r} e^{2 \pi in_r n_p/N} e^{2 \pi in_r n_q/N},
$$

we transform the estimator as

$$
P(k_F n_1) = \frac{V}{N^6 4 \pi s n_i^2 N^3} \sum_{n_r} \left[ \sum_{m_1 \geq n_1} \delta^{FFT}(m_1) e^{2 \pi in_r m_1/N} \right] \left[ \sum_{m_2 \geq n_1} \delta^{FFT}(m_2) e^{2 \pi in_r m_2/N} \right]
$$

Let’s define $\delta_{m_x}(n_r)$ as

$$
\delta_{m_x}(n_r) = \sum_{m \geq n_1} \delta^{FFT}(m) e^{2 \pi i m/N}
$$

In practice, $\delta_{m_x}(n_r)$ can be calculated by applying $\text{fftw\_dft\_z2r}$ to the array whose values are $\delta^{FFT}(m)$ when $|m - n_i| < s/2$ and otherwise zero. Note that $\delta_{m_x}(n_r)$ is real.
Then, we find the second estimator for the power spectrum as

\[ P(k_Fn_1) = \frac{V}{N^6} \frac{1}{4\pi m_1^3} \sum_{n_r} \delta_{n_1}(n_r)^2. \]  

(7.35)

This method takes more time than the first estimator which uses the direct sampling method. However, real strength of this method is apparent when we calculate the higher order polyspectra, e.g., Bispectrum and Trispectrum etc., because we do not have to explicitly sum up all the possible triangles (Bispectrum) or rectangles (Trispectrum), etc.

### 7.1.2.4 Estimating Bispectrum II

We can similarly construct the Bispectrum estimator. The direct sampling estimator we derived before is given by

\[ B(k_Fn_1, k_Fn_2, k_Fn_3) = \frac{V^2}{N^9} (\delta^{FFTW}(n_1)\delta^{FFTW}(n_2)\delta^{FFTW}(n_3))\delta^K(n_1 + n_2 + n_3), \]  

(7.36)

where we estimate the ensemble average by summing over the all possible triangles with side of \( k_1 - s k_F/2 < q_1 < k_1 + sk_F/2 \). Therefore, the estimator can be recasted as

\[ B(k_Fn_1, k_Fn_2, k_Fn_3) = \frac{V^2}{N^9} \sum_{m_1 \geq n_1} \sum_{m_2 \geq n_2} \sum_{m_3 \geq n_3} \delta^{FFTW}(m_1)\delta^{FFTW}(m_2)\delta^{FFTW}(m_3)\delta^K(m_{123}). \]  

(7.37)

Again, \( m_i \approx n_i \) means \( |m_i - n_i| < s/2 \). The denominator is the number of possible triangles with side of \( (k_1, k_2, k_3) \):

\[ \sum_{m_1 \geq n_1} \sum_{m_2 \geq n_2} \sum_{m_3 \geq n_3} \delta^D(m_{123}) = \frac{1}{V_F} \int_{k_1} d^3q_1 \int_{k_2} d^3q_2 \int_{k_3} d^3q_3 \delta^D(q_{123}) \]

\[ \approx \frac{8\pi^2 k_1 k_2 k_3 (\delta k)^3}{k_F^3} = \frac{8\pi^2 s^3 n_1 n_2 n_3}{k_F^3} \]  

(7.38)

By using the normalization of DFT,

\[ \delta^D(n_p + n_q + n_s) = \frac{1}{N^3} \sum_{n_t} e^{2\pi i n_r n_s/N} e^{2\pi i n_r n_s/N} e^{2\pi i n_r n_s/N}, \]

and following the exactly same procedure of finding the second estimator for power spectrum, we find the second bispectrum estimator

\[ B(k_Fn_1, k_Fn_2, k_Fn_3) = \frac{V^2}{N^9} \left( \frac{1}{8\pi^2 s^3 n_1 n_2 n_3} \right) \frac{1}{N^3} \sum_{n_r} \delta_{n_1}(n_r)\delta_{n_2}(n_r)\delta_{n_3}(n_r). \]  

(7.39)
7.1.2.5 Counting the number of triangles

In the previous section, we use the integral approximation to estimate the number of triangles. However, this approximation is broken down when three vectors are parallel to each other, i.e. when

\[ k_1 = \alpha k_2 = \beta k_3. \]

Also, it is not an accurate approximation for the triangle including the large scale modes. Therefore, in this section, we present the way we calculate the actual number of triangles for given triangular configurations.

We shall use the exactly same trick as you used in the previous section for estimating bispectrum. Since it is a trivial normalization factor, the number of triangles are the same as the bispectrum for the unit density contrast, i.e.

\[ N_{\text{tri}}(n_1, n_2, n_3) = \sum_{m_1 \geq n_1} \sum_{m_2 \geq n_2} \sum_{m_3 \geq n_3} \delta^K(m_1 + m_2 + m_3) \]

\[ = \frac{1}{N^3} \sum_{n_r} \left[ \sum_{m_1 \geq n_1} e^{2\pi i n_r \cdot m_1 / N} \right] \left[ \sum_{m_2 \geq n_2} e^{2\pi i n_r \cdot m_2 / N} \right] \left[ \sum_{m_3 \geq n_3} e^{2\pi i n_r \cdot m_3 / N} \right] \]

\[ = \frac{1}{N^3} \sum_{n_r} I_{n_1}(n_r) I_{n_2}(n_r) I_{n_3}(n_r), \]

where

\[ I_{n_i}(n_r) \equiv \sum_{m_i \geq n_i} e^{2\pi i n_r \cdot m_i / N}. \]

The function \( I_{n_i}(n_r) \) is the inverse Fourier transformation of the function in \( k \) space, which has the value unity within the shell of \( |m_i - n_i| < s / 2 \), otherwise zero.

By calculating the number of triangles, and compare the result with the exact calculation, we find that the logical size of the 1D Fourier grid has to be at least three times as large as that of the maximum wavenumber for which we want to estimate the bispectrum. That is, the array size of calculating \( I_{n_i}(n_r) \), \( bn_{\text{mesh}} \), has to satisfy

\[ bn_{\text{mesh}} > 3(s \times \text{nkmax}), \quad (7.40) \]

where \( s \) and \( \text{nkmax} \) are the bin size, and maximum wavenumber in the unit of the fundamental frequency. This is to avoid the fictitious increasing of the number of triangles (when calculating the number of triangles) or power (when calculating bispectrum) due to the aliasing effect.
7.1.3 Deconvolution

Now, we have the estimators for the power spectrum and the bispectrum. However, as we have employed the distribution scheme, the power spectrum and the bispectrum we would measure with those estimators are not the same as the power/bi-spectrum of the ‘real’ density contrast, but the power/bi-spectrum of density contrast convolved with the window function. Therefore, the power spectrum and bispectrum we estimate will show the artificial power suppression on small scales. Therefore, we have to deconvolve the window function due to the particle distribution scheme in order to estimate the power spectrum and bispectrum of the true density contrast.

7.1.3.1 Deconvolving only window function

First, as we know the exact shape of the window function in Fourier space, we can simply divide the resulting density contrast in Fourier space by the window function. That is, we deconvolve each $k$ mode of density contrast as

$$\delta(k) = \frac{\delta^m(k)}{W(k)},$$

(7.41)

or, deconvolve the estimated power spectrum by

$$P(k) = \left| \frac{\delta^m(k)}{W(k)} \right|^2 = P^m(k_1, k_2, k_3) \left[ \text{sinc} \left( \frac{\pi k_1}{2k_N} \right) \text{sinc} \left( \frac{\pi k_2}{2k_N} \right) \text{sinc} \left( \frac{\pi k_3}{2k_N} \right) \right]^{-2p},$$

(7.42)

for $k < k_N$. Again, $p = 1, 2, 3$ for NGP, CIC and TSC scheme, respectively. Here, superscript $m$ denote the measured quantity.

7.1.3.2 Deconvolving window function and aliasing

When we want to extract the power spectrum for $k \approx k_N$, we have to take the alias effect into account. As we show in Appendix A.2.1, the Fourier counterpart of the sampled data are the aliased sum of Fourier transformation of the continuous function. i.e.

$$P^m(k) = \sum_n |W(k + 2k_N n)|^2 P(k + 2k_N n)$$

(7.43)

For the case of uniformly distributed random particles, the power spectrum is simply a constant (Poisson shot noise), $1/n$, where $n$ is the number density. In that case, the

\footnote{Actually, it may not happen for most of N-body simulations because of the Force/Mass resolution is usually too poor to use the power spectrum for such a high wavenumber.}
The convolved power spectrum becomes

\[ P_m(k) = \frac{1}{n} \sum_n |W(k + 2k_N n)|^2. \]  \hspace{1cm} (7.44)

Let us calculate the convolved power spectrum analytically for \( p = 1, 2, 3 \) cases:

\[
\sum_n |W(k + 2k_N n)|^2 = 3 \prod_{i=1}^{p} \left[ \sum_{n=-\infty}^{\infty} W(k_i + 2k_N n)^2 \right] \\
= 3 \prod_{i=1}^{p} \left[ \sum_{n=-\infty}^{\infty} \frac{\sin^{2p} \left( \frac{\pi}{2k_N} (k_i + 2k_N n) \right)}{\left( \frac{\pi}{2k_N} (k_i + 2k_N n) \right)^{2p}} \right] \\
= 3 \prod_{i=1}^{p} \left[ \sin^{2p} \left( \frac{\pi k_i}{2k_N} \right) \sum_{n=-\infty}^{\infty} \left( \frac{\pi k_i}{2k_N} + \pi n \right)^{2p} \right] 
\]  \hspace{1cm} (7.45)

Here, the infinite summation can be calculated by using following identity (Jing, 2005). (so called Glaisher’s series \( ^2 \))

\[
\sum_{M=-\infty}^{\infty} \frac{1}{(a + Md)^2} = \left[ \frac{\pi}{d} \csc \frac{a \pi}{d} \right]^2 
\]  \hspace{1cm} (7.46)

**NGP (p=1)**

Let’s put \( d = \pi \).

\[
\sum_{n=-\infty}^{\infty} \frac{1}{(a + n\pi)^2} = \csc^2 a 
\]  \hspace{1cm} (7.47)

Therefore, for \( p = 1 \),

\[
\sum_n |W(k + 2k_N n)|^2 = 3 \prod_{i=1}^{p} \sin^{2p} \left( \frac{\pi k_i}{2k_N} \right) \csc^2 \left( \frac{\pi k_i}{2k_N} \right) = 1, 
\]  \hspace{1cm} (7.48)

and sum over infinite aliasing effect of power suppression ends up giving rise to the original power.

**CIC (p=2)**

Differentiate both side with respect to \( a \) twice, we get

\[
\sum_{n=-\infty}^{\infty} \frac{6}{(a + n\pi)^4} = 2(3 - 2\sin^2 a) \csc^4 a 
\]  \hspace{1cm} (7.49)

Therefore, the aliased sum becomes

$$\sum_n |W(k + 2k_N n)|^2 = \prod_{i=1}^{3} \sin^4 \left( \frac{\pi k_i}{2k_N} \right) \left[ 1 - \frac{2}{3} \sin^2 \left( \frac{\pi k_i}{2k_N} \right) \right] \csc^4 \left( \frac{\pi k_i}{2k_N} \right)$$

$$= \prod_{i=1}^{3} \left[ 1 - \frac{2}{3} \sin^2 \left( \frac{\pi k_i}{2k_N} \right) \right]. \quad (7.50)$$

**TSC (p=3)**

Differentiate the both side with a four times, we get

$$\sum_{n=-\infty}^{\infty} \frac{120}{(a + n\pi)^6} = (120 - 120 \sin^2 a + 16 \sin^4 a) \csc^6 a \quad (7.51)$$

Therefore,

$$\sum_n |W(k + 2k_N n)|^2$$

$$= \prod_{i=1}^{3} \sin^6 \left( \frac{\pi k_i}{2k_N} \right) \left[ 1 - \sin^2 \left( \frac{\pi k_i}{2k_N} \right) + \frac{2}{15} \sin^4 \left( \frac{\pi k_i}{2k_N} \right) \right] \csc^6 \left( \frac{\pi k_i}{2k_N} \right)$$

$$= \prod_{i=1}^{3} \left[ 1 - \sin^2 \left( \frac{\pi k_i}{2k_N} \right) + \frac{2}{15} \sin^4 \left( \frac{\pi k_i}{2k_N} \right) \right]. \quad (7.52)$$

When the measured power spectrum include the both true power spectrum and Poisson shot noise, we have to cure it iteratively, See, e.g. Jing (2005). However, as the nonlinear galaxy power spectrum is dominated by the constant term $P_0$ and large $k$ plateau of nonlinear bias terms, $P_{b2}$, $P_{b22}$, we may simply apply the same aliased window function as for the constant power spectrum. In practice, we calculated both window-corrected and window-alias-corrected power spectrum, and only use the wavenumber ranges where those two power spectra agree with each other.

### 7.1.4 Poisson shot noise

Finally, we have to subtract the Poisson shot noise from the deconvolved power spectrum and bispectrum. As a general terminology, ‘shot noise’ refers to the self-particle contribution to the statistics. We call it Poisson shot noise, as it appears when we think of a realization of galaxy (matter) distribution as a Poisson sampling of underlying smooth galaxy (matter) density contrast field. Poisson shot noise appears whenever we calculate the n-point function from a discrete set of objects, like a galaxy, dark matter, etc.
7.1.4.1 Poisson sampling and underlying density field

Suppose we have a point process \( n(r) \) which is a “Poisson sample” of some continuous stochastic field \( 1 + \delta(r) \). That is, the probability that an infinitesimal volume element \( \delta V \) contains an object is \( \bar{n}(r) \left[ 1 + \delta(r) \right] \delta V \).

Following Peebles (1980, §36) we describe the process by dividing the space into the infinitesimal micro-cells of volume \( \delta V \) which has a occupation numbers \( n_i = 0 \) or \( 1 \). That is, the statistical average of self-correlator for a given cell is

\[
\langle n_i^2 \rangle = \langle n_i \rangle = \bar{n}(r) \delta V,
\]

and the correlator for different cells are given by the underlying density contrast as

\[
\langle n_i n_j \rangle_{i \neq j} = \bar{n}(r) \bar{n}(r') \delta V_i \delta V_j \left[ 1 + \langle \delta(r) \delta(r') \rangle \right]
\]

\[
\langle n_i n_j n_k \rangle_{i \neq j \neq k} = \bar{n}(r) \bar{n}(r) \bar{n}(r') \delta V_i \delta V_j \delta V_k \times \left[ 1 + \langle \delta(r) \delta(r') \rangle + \langle \delta(r) \delta(r) \rangle + \langle \delta(r') \delta(r) \rangle + \langle \delta(r) \delta(r') \rangle \right].
\]

7.1.4.2 Power spectrum and Bispectrum of discrete particles

We follow the derivation given in Feldman et al. (1994). Consider the expectation value of

\[
\int d^3r \int d^3r' g(r, r') n(r) n(r'),
\]

for an arbitrary function \( g(r, r') \). By using the infinitesimal micro-cells, the expectation value becomes

\[
\left\langle \int d^3r \int d^3r' g(r, r') n(r) n(r') \right\rangle = \int d^3r \int d^3r' g(r, r') \langle n(r) n(r') \rangle = \sum_{i,j} g(r_i, r_j) \langle n_i n_j \rangle
\]

\[
= \sum_{i \neq j} \delta V_i \delta V_j g(r_i, r_j) \bar{n}(r_i) \bar{n}(r_j) \left[ 1 + \langle \delta(r) \delta(r') \rangle \right] + \sum_{i \neq j} \delta V_i g(r_i, r_i) \bar{n}(r_i)
\]

\[
= \int d^3r \int d^3r' g(r, r') \bar{n}(r') \left[ 1 + \langle \delta(r) \delta(r') \rangle \right] + \int d^3r g(r, r) \bar{n}(r)
\]

\[
= \int d^3r \int d^3r' g(r, r') \left\{ \bar{n}(r) \bar{n}(r') \left[ 1 + \langle \delta(r) \delta(r') \rangle \right] + \bar{n}(r) \delta^D(r - r') \right\}.
\]
As this equation holds for an arbitrary function \( g(r, r') \), comparing the first and last line of the equation above, we find the relation between the two point correlator of the discrete number density and that of underlying density contrast:

\[
\langle n(r)n(r') \rangle = \bar{n}(r)\bar{n}(r')[1 + \langle \delta(r)\delta(r') \rangle] + \bar{n}(r)\delta^D(r - r'). \tag{7.57}
\]

The 2nd term in Equation (7.57) is called a \textit{Poisson shot noise}, as the Dirac delta function manifests its identity as a self-particle contribution. The shot noise term \( \bar{n}(r)\delta^D(r - r') \) can also be understood as follows. Because the presence of galaxies obeys the Poisson statistics, for a single position \( r \) with the mean number density \( \bar{n}(r) \), its variance is \( \sigma^2[n(r)] \equiv \langle n(r)n(r) \rangle - \bar{n}(r)^2 = \bar{n}(r) \).

By using equation (7.57), the correlation function of galaxy density contrast measured from the ‘discrete’ samples of galaxies is given by

\[
\xi_n(r - r') \equiv \langle \delta_n(r)\delta_n(r') \rangle = \langle \left( \frac{n(r) - \bar{n}(r)}{\bar{n}(r')} \right) \left( \frac{n(r') - \bar{n}(r)}{\bar{n}(r)} \right) \rangle = \frac{\langle [n(r)n(r') - \bar{n}(r)\bar{n}(r')] \rangle}{\bar{n}(r)\bar{n}(r')} = \langle \delta(r)\delta(r') \rangle + \frac{\delta^D(r - r')}{\bar{n}(r)}, \tag{7.58}
\]

where we denote the density contrast of the discrete Poisson sample as

\[
\delta_n(r) \equiv \frac{n(r) - \bar{n}(r)}{\bar{n}(r)}.
\]

The Fourier transform of Equation (7.58) yields following relation in Fourier space:

\[
\langle \delta_n(k)\delta_n(k') \rangle = \langle \delta(k)\delta(k') \rangle + \frac{1}{\bar{n}(r)} \int d^3r e^{-ir\cdot(k+k')} \delta^D(\bar{n}(r)). \tag{7.59}
\]

For N-body simulation, as the mean number density over the simulation box are constant, \( \bar{n}(r) \equiv \bar{n} \); the equation (7.59) reduces to

\[
\langle \delta_n(k)\delta_n(k') \rangle = \langle \delta(k)\delta(k') \rangle + \frac{(2\pi)^3}{\bar{n}} \delta^D(k' + k). \tag{7.60}
\]

Therefore, we get the formula for Poisson shot noise.

\[
P_n(k) = P(k) + \frac{1}{\bar{n}} \tag{7.61}
\]
7.1.4.3 Bispectrum of discrete particles

Similarly, we calculate the bispectrum of discrete particles from the expectation value of
\[
\int d^3 r_1 \int d^3 r_2 \int d^3 r_3 g(r_1, r_2, r_3) n(r_1) n(r_2) n(r_3)
\]
for an arbitrary function \( g(r_1, r_2, r_3) \). By using the infinitesimal micro-cells, the expectation value becomes
\[
\left\langle \int d^3 r_1 \int d^3 r_2 \int d^3 r_3 g(r_1, r_2, r_3) n(r_1) n(r_2) n(r_3) \right\rangle
= \int d^3 r_1 \int d^3 r_2 \int d^3 r_3 g(r_1, r_2, r_3) \left\langle n(r_1) n(r_2) n(r_3) \right\rangle
= \sum_{i,j,k} g(i, j, k) \langle n n n \rangle
= \sum_{i,j,k} \delta \bar{V}_i \delta \bar{V}_j \delta \bar{V}_k \bar{n}_i \bar{n}_j \bar{n}_k [1 + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle]
+ \sum_{i=j=k} \delta \bar{V}_i \delta \bar{V}_i \delta \bar{V}_i \bar{n}_i [1 + \langle \delta \delta \delta \rangle + \langle 2 \text{ cyclic} \rangle]
+ \sum_{i=j=k} \delta \bar{V}_i \delta \bar{V}_i \delta \bar{V}_i \bar{n}_i [1 + \langle \delta \delta \delta \rangle + \langle 2 \text{ cyclic} \rangle]
= \int d^3 r_1 \int d^3 r_2 \int d^3 r_3 g(r_1, r_2, r_3)
\times \left\{ \langle \bar{n}(r_1) \bar{n}(r_2) \bar{n}(r_3) \rangle [1 + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle] + \langle 2 \text{ cyclic} \rangle \right\}
+ \langle \bar{n}(r_1) \delta \bar{D}(r_1 - r_2) \delta \bar{D}(r_1 - r_3) \rangle \right\},
\tag{7.62}
\end{equation}
where we use subscript to mark the coordinate in the fourth and fifth line. For example, \( g_{ijk} \equiv g(r_i, r_j, r_k) \). Again, as the equation (7.62) has to hold for arbitrary \( g(r_1, r_2, r_3) \), we find
\[
\left\langle n(r_1) n(r_2) n(r_3) \right\rangle
= \bar{n}(r_1) \bar{n}(r_2) \bar{n}(r_3) [1 + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle] + \langle 2 \text{ cyclic} \rangle
+ \bar{n}(r_1) \bar{n}(r_2) \bar{n}(r_3) [1 + \langle \delta \delta \delta \rangle + \langle \delta \delta \delta \rangle] + \langle 2 \text{ cyclic} \rangle
+ \bar{n}(r_1) \delta \bar{D}(r_1 - r_2) \delta \bar{D}(r_1 - r_3). \tag{7.63}
\]

We calculate the three-point correlation function of discrete particles by
\[
\left\langle \delta_n(r_1) \delta_n(r_2) \delta_n(r_3) \right\rangle = \left\langle \left( \frac{n(r_1)}{\bar{n}(r_1)} - 1 \right) \left( \frac{n(r_2)}{\bar{n}(r_2)} - 1 \right) \left( \frac{n(r_3)}{\bar{n}(r_3)} - 1 \right) \right\rangle
= \frac{\langle n(r_1) n(r_2) n(r_3) \rangle}{\bar{n}(r_1) \bar{n}(r_2) \bar{n}(r_3)} - \frac{\langle n(r_1) n(r_2) \rangle}{\bar{n}(r_1) \bar{n}(r_2)} + (2 \text{ cyclic}) + 2. \tag{7.64}
\]

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Using equation (7.57) and equation (7.63), we find
\[
\langle \delta_n(r_1)\delta_n(r_2)\delta_n(r_3) \rangle \\
= \langle \delta(r_1)\delta(r_2)\delta(r_3) \rangle + \frac{\delta^D(r_1 - r_3)}{\bar{n}(r_3)}\langle \delta(r_1)\delta(r_2) \rangle + (2 \text{ cyclic}) \\
+ \frac{1}{\bar{n}(r_2)\bar{n}(r_3)}\delta^D(r_1 - r_2)\delta^D(r_1 - r_3),
\] (7.65)

and the Fourier transform of equation (7.65) yields,
\[
\langle \delta_n(k_1)\delta_n(k_2)\delta_n(k_3) \rangle = \langle \delta(k_1)\delta(k_2)\delta(k_3) \rangle \\
+ \int d^3r_1 \int d^3r_2 \frac{1}{\bar{n}(r_1)} \langle \delta(r_1)\delta(r_2) \rangle e^{-ir_1 \cdot (k_1 + k_3) - ir_2 \cdot k_2} \\
+ \int d^3r_1 \frac{1}{\bar{n}^2(r_1)} e^{-ir_1 \cdot k_{123}},
\] (7.66)

Especially, when the mean number density does not vary in time, the three point function in $k$ space becomes
\[
\langle \delta_n(k_1)\delta_n(k_2)\delta_n(k_3) \rangle \\
= \langle \delta(k_1)\delta(k_2)\delta(k_3) \rangle + \frac{1}{\bar{n}} \langle \delta(k_1)\delta(k_2) \rangle + (2 \text{ cyclic}) + \frac{(2\pi)^3}{\bar{n}^2}\delta^D(k_{123}),
\] (7.67)

and, the bispectrum of discrete Poisson samples is reduced to
\[
B_n(k_1, k_2, k_3) = B(k_1, k_2, k_3) + \frac{1}{\bar{n}} \left\{ P(k_1) + P(k_2) + P(k_3) \right\} + \frac{1}{\bar{n}^2}.
\] (7.68)

### 7.2 Power spectrum from Galaxy surveys

For the real galaxy survey, we have to take into account the spatial variance of the mean number density. In that case, we have to weight galaxies differently depending on the mean number density in order to optimize the variance of power spectrum. The most popular weighting is given from Feldman et al. (1994, FKP). In this section, we first review the FKP estimator, and derive the optimal weighting function for power spectrum measurement. Then, we show the practical implementation of FKP estimator for galaxy surveys by using Discrete Fourier Transformation.
7.2.1 The FKP estimator

7.2.1.1 The power spectrum with weighting function \( w(r) \)

Let us denote the weighted overdensity \( F(r) \) as a ‘weighted’ overdensity of a galaxies at position \( r \) in the survey:

\[
F(r) \equiv w(r) [n(r) - \bar{n}(r)] = w(r)\bar{n}(r)\delta_n(r) \equiv W(r)\delta_n(r).
\]

(7.69)

Here, \( \bar{n}(r) \) is the mean number density of galaxies expected at position \( r \), and \( w(r) \) is a weighting function which optimizes the variance of the estimated power spectrum. Combining the two effect of the selection function and the weighting defines the window function, \( W(r) \).

By convolution theorem, the Fourier mode of the weighted density field becomes

\[
F(k) = \int \frac{d^3q}{(2\pi)^3} W(k - q)\delta_n(q).
\]

(7.70)

The two point correlation function of the weighted density field is

\[
\langle F(k)F(k') \rangle = \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} W(k - q)W(k' - q')\langle \delta_n(q)\delta_n(q') \rangle
\]

\[
= \int \frac{d^3q}{(2\pi)^3} W(k - q)W(k' + q)P(q)
\]

\[
+ \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} W(k - q)W(k' - q') \int d^3r \frac{1}{\bar{n}(r)}e^{-i\mathbf{r}(q+q')}.
\]

The second term is further simplifies to

\[
\int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} W(k - q)W(k' + q') \int d^3r \frac{1}{\bar{n}(r)}e^{-i\mathbf{r}(q+q')}
\]

\[
= \int d^3r W^2(r)\frac{1}{\bar{n}(r)}e^{-i\mathbf{r}(k+k')} = \int d^3r w^2(r)\bar{n}(r)e^{-i\mathbf{r}(k+k')}.
\]

Therefore, the power spectrum of the weighted overdensity can be calculated from

\[
\langle F(k)F(k') \rangle = \int \frac{d^3q}{(2\pi)^3} W(k - q)W(k' + q)P(q) + \int d^3r w^2(r)\bar{n}(r)e^{-i\mathbf{r}(k+k')},
\]

(7.71)

that is,

\[
\langle F(k)F(-k) \rangle = \int \frac{d^3q}{(2\pi)^3} |W(k - q)|^2 P(q) + \int d^3r w^2(r)\bar{n}(r).
\]

(7.72)

In Feldman, Kaiser & Peacock (1994, FKP), they first subtract the synthetic catalog, \( n_s(r) \), generated by the number density of \( \bar{n}(r)/\alpha \). Also, they divide the ‘magnitude’
of the window function

\[ W \equiv \left[ \int d^3r W^2(r) \right]^{1/2} = \left[ \frac{\int d^3k}{(2\pi)^3} |W(k)|^2 \right]^{1/2} \]

when defining the ‘weighted’ density field. That is, the weighted density field in FKP is

\[ F(r) \equiv \frac{w(r) [n(r) - \alpha s(r)]}{W} = \frac{W(r)}{W} [\delta_n(r) - \delta_s(r)], \quad (7.73) \]

where

\[ \delta_s(r) \equiv \frac{n_s(r) - \bar{n}(r)/\alpha}{\bar{n}(r)/\alpha}. \]

Assuming that the synthetic catalog is completely random, and independent of the galaxy distribution, i.e.,

\[ \langle \delta_n(r) \delta_n(r') \rangle = 0 \quad (7.74) \]
\[ \langle \delta_s(r) \delta_s(r') \rangle = \alpha \frac{\delta^D(r-r')}{\bar{n}(r)}, \quad (7.75) \]

the power spectrum of the weighted density field of FKP is

\[ \tilde{P}(k) \equiv \left\langle |F(k)|^2 \right\rangle = \int \frac{d^3q}{(2\pi)^3} \frac{|W(k-q)|^2}{W^2} P(q) + \frac{1 + \alpha}{W^2} \int d^3r w^2(r) \bar{n}(r). \quad (7.76) \]

Note that in FKP, they refer \( W(k)/W \) as \( G(k) \):

\[ G(k) \equiv \frac{\int d^3r \bar{n}(r) w(r) e^{-ik \cdot r}}{\left[ \int d^3r \bar{n}^2(r) w^2(r) \right]^{1/2}}. \]

If survey has a typical size of \( D \simeq V^{1/3} \), then the window function \( W(k)/W \) will be a rather compact function with width \( \delta k \sim 1/D \). Therefore, for high enough wavenumber, \( |k| \gg 1/D \), we can approximate the power spectrum of weighted overdensity as

\[ \tilde{P}(k) \simeq P(k) + P_{\text{shot}} \]

provided that \( P(k) \) is a smooth function of \( k \). Here,

\[ P_{\text{shot}} \equiv \frac{1 + \alpha}{W^2} \int d^3r \bar{n}(r) w^2(r). \]

is the shot noise power spectrum in the presence of the weighting function. We estimate the angular averaged (monopole) power spectrum estimation by

\[ \hat{P}(k) \equiv \frac{1}{V_k} \int_{V_k} d^3q \left[ \tilde{P}(q) - P_{\text{shot}} \right]. \quad (7.77) \]

where \( V_k \) is the volume of the shell in Fourier space.
7.2.2 The variance of the power spectrum

We estimate the variance of the power spectrum by

$$\sigma_p^2 \equiv \left\langle \left[ \delta \hat{P}(q) - P(q) \right]^2 \right\rangle = \frac{1}{V_k} \int d^3q \int d^3q' \left\langle \delta \hat{P}(q) \delta \hat{P}(q') \right\rangle$$  \hspace{1cm} (7.78)

with $\delta \hat{P}(q) = \hat{P}(q) - P(q)$. If $F(k)$ obeys Gaussian statistics, then

$$\left\langle \delta \hat{P}(q) \delta \hat{P}(q') \right\rangle = |\langle F(q) F^*(q') \rangle|^2$$

With Equation (7.71), we write the right hand side as

$$\langle F(q) F^*(q') \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{W(q-p)W^*(q'-p)}{W^2} P(p) + \frac{1 + \alpha}{W^2} \int d^3r w^2(r)\bar{n}(r)e^{-ir(q-q')}.$$  \hspace{1cm} (7.79)

In FKP paper, the second term is referred as $S(k)$:

$$S(k) \equiv \frac{(1 + \alpha) \int d^3r w^2(r)\bar{n}(r)e^{-ikr}}{\int d^3r w^2(r)\bar{n}^2(r)}$$

As Equation (7.78) integrate over the same spherical shell, we consider the case when $q-q' = \delta q$. For $|q| \gg 1/D$, we approximate Equation (7.79) as

$$\langle F(q) F^*(q') \rangle \approx \frac{P(q)}{W^2} \int \frac{d^3p}{(2\pi)^3} W(q-p)W^*(q'-p) + \frac{1 + \alpha}{W^2} \int d^3r w^2(r)\bar{n}(r)e^{-ir(q-q')}.$$  \hspace{1cm} (7.80)

where the first term becomes

$$\int \frac{d^3p}{(2\pi)^3} W(q-p)W^*(q'-p) = \int \frac{d^3p}{(2\pi)^3} \int d^3r \int d^3r' W(r)W(r')e^{-ir-p}e^{ir'}(q-p)$$

$$= \int d^3r \int d^3r' W(r)W(r')\delta^3(r-r')e^{ir'}q$$

$$= \int d^3r W^2(r)e^{-ir(q-q')}.$$  \hspace{1cm} (7.81)

In FKP, this function is called $Q(k)$:

$$Q(k) \equiv \int \frac{d^3r w^2(r)\bar{n}^2(r)e^{-ikr}}{\int d^3r w^2(r)\bar{n}^2(r)}.$$  \hspace{1cm} (7.81)

Therefore, the variance of the power spectrum is

$$\left\langle \delta \hat{P}(q) \delta \hat{P}(q') \right\rangle = |\langle F(q) F^*(q') \rangle|^2$$

$$= |P(q)Q(\delta q) + S(\delta q)|^2.$$  \hspace{1cm} (7.82)
7.2.3 Optimal weighting

When the spherical shell is larger than the coherence length (~ 1/D), the double integration in Equation (7.78) reduces to

\[ \sigma_p^2(k) = \frac{1}{V_k} \int d^3q |P(k)Q(q) + S(q)|^2. \]

Therefore, the fractional variance of the power is

\[ \frac{\sigma_p^2(k)}{P^2(k)} = \frac{1}{V_k} \int d^3q \left| Q(q) + \frac{S(q)}{P(k)} \right|^2 \]

\[ = \frac{1}{V_k W} \int d^3q \left| d^3r w^2(r) \bar{n}^2(r) \left( 1 + \frac{1 + \alpha}{\bar{n}(r)P(k)} \right) e^{-i\bar{q} \cdot r} \right|^2. \quad (7.83) \]

Using Parseval’s theorem, the equation is further simplified as

\[ \frac{\sigma_p^2(k)}{P^2(k)} = \frac{(2\pi)^3}{V_k W^2} \int d^3q w^4(r) \bar{n}^4(r) \left( 1 + \frac{1 + \alpha}{\bar{n}(r)P(k)} \right)^2. \quad (7.84) \]

For optimal weighting, we choose the weighting function \( w(r) \) which minimizes the variance in Equation (7.84). First, let’s abbreviate the equation as

\[ \frac{\sigma_p^2(k)}{P^2(k)} = \frac{\int d^3q w^4(r) f(r)}{\left[ \int d^3q w^2(r) g(r) \right]^2} \]

That is,

\[ f(r) = \bar{n}^4(r) \left( 1 + \frac{1 + \alpha}{\bar{n}(r)P(k)} \right)^2 \]

\[ g(r) = \bar{n}^2(r). \]

When we take the variation of \( w(r) = w_0(r) + \delta w(r) \), Equation (7.84) becomes

\[ \frac{\sigma_p^2(k)}{P^2(k)} = \frac{\int d^3q w_0^4 \left[ 1 + 4\delta w/w_0 \right] f}{\left[ \int d^3q w_0^2 \left( 1 + 3\delta w/w_0 \right) g \right]^2} \]

\[ = \frac{\int d^3q w_0^4 f}{\left[ \int d^3q w_0^2 g \right]^2} \left\{ 1 + 4 \left( \frac{\int d^3qw_0^2\delta w f}{\int d^3qw_0^4 f} + \frac{\int d^3qw_0\delta w g}{\int d^3qw_0^4 g} \right) \right\} + \cdots \quad (7.85) \]

3The proof of Parseval’s theorem is given as following.

\[ \int d^3q \int d^3r f(r) e^{-i\bar{q} \cdot r} = \int d^3q \int d^3r f(r)e^{-i\bar{q} \cdot r} \int d^3r' f(r')e^{i\bar{q} \cdot r'} \]

\[ = (2\pi)^3 \int d^3r \int d^3r' f(r)f(r') \delta^3(r - r') = (2\pi)^3 \int d^3r f^2(r). \]
Therefore, the optimal weighting function has to satisfy
\[
\int d^3r w_0^2 \delta w f = \int d^3r w_0^2 \delta w g,
\]
whose solution is given by
\[
w_0^2 \propto g/f.
\]
In terms of the mean number density and the power spectrum, the optimal weighting has to satisfy
\[
w(r) \propto \frac{1}{\bar{n}(r) + (1 + \alpha)/P(k)}.
\]
Finally, the dimensionless optimal weighting function is
\[
w(r) = \frac{1}{1 + \alpha + P(k)\bar{n}(r)}.
\]

When we choose a large number of synthetic sample (\(\alpha \ll 1\)), we recover the result of the FKP optimal weighting function:
\[
w(r) = \frac{1}{1 + P(k)\bar{n}(r)}.
\]

7.3 Implementing the FKP estimator

In this section, we show the Discrete Fourier Transform implementation of the FKP estimator. Let us consider the case when the mean number density depends on the position: \(\bar{n}(r)\). In the previous section, we have shown that the optimal weighting function is given by
\[
w(r) = \frac{1}{1 + \bar{n}(r)P(k)}.
\]
We define the weighted overdensity in the discrete grid point \(n_r\) as
\[
F(n_r) = \frac{w(n_r)}{W} [N_g(n_r) - \alpha N_s(n_r)] = \frac{w(n_r)\bar{N}(n_r)}{W} [\delta_g(n_r) - \delta_s(n_r)]
\]
\[
= \frac{W(n_r)H^3}{W} [\delta_g(n_r) - \delta_s(n_r)].
\]
Here, \(N_i(n_r)\) denote the number density of galaxies (\(i = g\)), and synthetic random samples (\(i = s\)) assigned to the grid \(n_r\),
\[
N_i(n_r) \equiv \int_{n_r} d^3r n_i(r) W^3(r - n_r),
\]
where $W^s(x)$ is the sampling window function we discussed in Section 7.1.1. Similarly, $\bar{N}(n_r)$ is the mean number density of galaxies assigned to the grid $n_r$:

$$\bar{N}(n_r) \equiv \int_{n_r} d^3r \bar{n}(r) W^s(r-n_r).$$

(7.94)

As synthetic random samples are generated from the mean number density rescaled by $1/\alpha$, they are related by

$$\bar{N}(n_r) = \langle N_g(n_r) \rangle = \alpha \langle N_s(n_r) \rangle.$$

(7.95)

Also we define the survey window function on the grid point $n_r$ as

$$W(n_r) \equiv \frac{w(n_r)\bar{N}(n_r)}{H^3}.$$

(7.96)

Fourier transform the weighted overdensity above yields

$$F_{DFT}(n_k) = \sum_{n_r} F(n_r) e^{-i2\pi n_r \cdot n_r/N} = \frac{H^3}{W} \sum_{n_r} W(n_r) [\delta_n(n_r) - \delta_s(n_r)] e^{-i2\pi n_r \cdot n_r/N}. \quad (7.97)$$

That is, the Fourier transform is given by the convolution of $W(n_k)$ and $\delta_n(n_k) - \delta_s(n_k)$. Let us be explicit about the convolution in DFT. For discrete sampling of $A(n_r)$ and $B(n_r)$, the Fourier transform of its multiplication $C(n_r) \equiv A(n_r)B(n_r)$ is given by

$$C_{DFT}(n_k) \equiv A(n_r)B(n_r)e^{-i2\pi n_r \cdot n_r/N}
= \sum_{n_r} \left[ \frac{1}{V} \sum_{n_p} A(n_p)e^{i2\pi n_p \cdot n_r/N} \right] \left[ \frac{1}{V} \sum_{n_q} B(n_q)e^{i2\pi n_q \cdot n_r/N} \right] e^{-i2\pi n_k \cdot n_r/N}
= \frac{1}{V^2} \sum_{n_p} \sum_{n_q} A(n_p)B(n_q) e^{i2\pi (n_p + n_q) \cdot n_r/N} e^{-i2\pi n_k \cdot n_r/N}
= \frac{N^3}{V^2} \sum_{n_q} A(n_q)B(n_q) \delta_{n_k \pm mN \cdot n_q \pm n_r}
= \frac{N^3}{V^2} \sum_{n_q} A((n_k - n_q)_N)B(n_q). \quad (7.98)$$

---

*For the normalization of the Discrete Fourier Transform, see Appendix A.*

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Here, \(m\) can be any integer, and \(A((n_k)_N)\) denotes the integer triplet modulated by \(N\). Using the convolution in DFT, we find the DTF of \(F\) becomes

\[
F_{\text{DFT}}(n_k) = \frac{1}{WV} \sum_{n_q} W((n_k - n_q)_N) [\delta_n(n_q) - \delta_s(n_q)].
\]

(7.99)

Note that when \(W(n_q) \equiv \bar{w}_n\) (constant), then,

\[
W(n_k) = H^3 \sum_{n_q} \bar{w}_n e^{-i2\pi n_q \cdot n_k/N} = H^3 \bar{w}_n N^3 \delta_{n_k, 0}
\]

therefore, we reproduce the normalization of the previous section:

\[
F_{\text{DFT}}(n_k) = \frac{\bar{w}_n}{W} [\delta_n(n_k) - \delta_s(n_k)].
\]

We estimate the power spectrum by calculating the ensemble average of

\[
\langle F_{\text{DFT}}(n_k) F_{\text{DFT}}(n_{k'}) \rangle = \frac{1}{W^2V^2} \sum_{n_q} \sum_{n_q'} W((n_k - n_q)_N)W((n_{k'} - n_{q'})_N)
\]

\[
\times \langle [\delta_n(n_q) - \delta_s(n_q)] [\delta_n(n_{q'}) - \delta_s(n_{q'})] \rangle.
\]

(7.100)

First, let us evaluate the two point correlation function without the window function for the discrete grid

\[
\langle \delta_n(n_q) \delta_s(n_{q'}) \rangle = H^6 \sum_{n_r} \sum_{n_r'} \langle \delta_n(n_q) \delta_s(n_{q'}) \rangle e^{-i2\pi n_q \cdot n_r/N} e^{-i2\pi n_{q'} \cdot n_r/N}
\]

\[
= H^6 \sum_{n_r} \sum_{n_r'} \left[ \langle \delta(n_q) \delta(n_{q'}) \rangle + \frac{\delta_k}{H^3 \bar{n}(n_q)} \right] e^{-i2\pi n_q \cdot n_r/N} e^{-i2\pi n_{q'} \cdot n_r/N}
\]

\[
= \langle \delta(n_q) \delta(n_{q'}) \rangle + H^3 \sum_{n_r} \frac{1}{\bar{n}(n_q)} e^{-i2\pi n_q \cdot (n_q + n_{q'})/N},
\]

(7.101)

where in the second line, we changes the Dirac delta function to the Kronecker delta by explicitly factoring out the dimensionality \((1/H^3)\). From the same procedure, we can also calculate

\[
\langle \delta_n(n_q) \delta_s(n_{q'}) \rangle = 0,
\]

(7.102)

\[
\langle \delta_s(n_q) \delta_s(n_{q'}) \rangle = H^3 \sum_{n_r} \frac{\alpha}{\bar{n}(n_q)} e^{-i2\pi n_q \cdot (n_q + n_{q'})/N}.
\]

(7.103)
Putting the result above all together, we finally have

\[
\langle F_{DFT}(\mathbf{n}_k)F_{DFT}(\mathbf{n}_{k'}) \rangle = \frac{1}{W^2 V^2} \sum_{n_k} \sum_{n_{k'}} W((\mathbf{n}_k - \mathbf{n}_q)_N)W((\mathbf{n}_{k'} - \mathbf{n}_{q'})_N) \langle \delta(\mathbf{n}_q) \delta(\mathbf{n}_{q'}) \rangle 
\]

\[
+ H^2 \frac{1 + \alpha}{W^2 V^2} \sum_{n_k} \sum_{n_{k'}} W((\mathbf{n}_k - \mathbf{n}_q)_N)W((\mathbf{n}_{k'} - \mathbf{n}_{q'})_N) \sum_{n_r} \frac{1}{\tilde{n}(n_r)} e^{-i2\pi n_r (n_k + n_{k'})/N}. 
\]

(7.104)

The first term is the Window function convolved galaxy power spectrum:

\[
\frac{1}{W^2 V^2} \sum_{n_k} \sum_{n_{k'}} W((\mathbf{n}_k - \mathbf{n}_q)_N)W((\mathbf{n}_{k'} - \mathbf{n}_{q'})_N) \frac{(2\pi)^3}{k_F1k_F2k_F3} P(n_q) \delta^D_{n_q,n_{q'}} 
\]

\[
= \frac{1}{W^2 V^2} \sum_{n_k} W((\mathbf{n}_k - \mathbf{n}_q)_N)W((\mathbf{n}_k + \mathbf{n}_q)_N) P(n_q). 
\]

(7.105)

and the second term is the shot-noise term:

\[
H^2 \frac{1 + \alpha}{W^2 V^2} \sum_{n_k} \sum_{n_{k'}} \sum_{n_{1}} \sum_{n_{2}} W(n_{1})e^{-i2\pi n_{13} (n_k - n_q)/N} 
\times \sum_{n_{2}} W(n_{2})e^{-i2\pi n_{23} (n_{1} - n_{q'})/N} \sum_{n_r} \frac{1}{\tilde{n}(n_r)} e^{-i2\pi n_r (n_k + n_{q'})/N} 
\]

\[
= H^2 \frac{1 + \alpha}{W^2 V^2} \sum_{n_{1}} \sum_{n_{2}} \sum_{n_r} \frac{W(n_{1})W(n_{2})}{\tilde{n}(n_r)} e^{-i2\pi n_{13} (n_k - n_{q'})/N} N^3 \delta^D_{n_{13},n_{23}} N^3 \delta^D_{n_{23},n_{13}} 
\]

\[
= H^2 \frac{1 + \alpha}{W^2 V^2} \sum_{n_r} \frac{W^2(n_r)}{\tilde{n}(n_r)} e^{-i2\pi n_r (n_k + n_{q'})/N}. 
\]

(7.106)

Adding up the results, we find that

\[
\langle |F_{DFT}(\mathbf{n}_k)|^2 \rangle = \frac{1}{W^2 V} \sum_{n_q} W((\mathbf{n}_k - \mathbf{n}_q)_N)^2 P(n_q) + H^2 \frac{1 + \alpha}{W^2} \sum_{n_r} \frac{W^2(n_r)}{\tilde{n}(n_r)}. 
\]

(7.107)

The normalization factor \( W \) can be calculated as

\[
W \equiv \left[ \int d^3 r W^2(r) \right]^{1/2} = \left[ \sum_{n_r} \frac{w^2(n_r)N^2(n_r)}{H^3} \right]^{1/2}. 
\]

(7.108)

Let us check the limiting case when \( \tilde{n}(n_r) = \bar{n} \). For that case, the weighting function
is also a constant, \( w(n_r) = w \). Then, the normalization factor becomes

\[
W = \left[ \sum_{n_r} w^2(n_r) H^3 \right]^{1/2} = \left[ N^3 w^2(n_r) H^3 \right]^{1/2} = \sqrt{V} w(n)
\]  (7.109)

and power spectrum can be estimated by

\[
\langle |F_{DFT}(n_k)|^2 \rangle = \frac{1}{V^2 w^2 H^2} \sum_{n_q} |W(n_k - n_q)|^2 \tilde{P}(n_q) + H^3 \frac{1 + \alpha}{V} \sum_{n_r} \frac{w^2(n_r)}{n_r}
\]

\[
= \tilde{P}(n_k) + \frac{1 + \alpha}{n}
\]  (7.110)

which are what we expected from the calculation of the previous section.

Finally, we have to correct for the window function due to the number density distribution, by following the method described in Section 7.1.3.

### 7.3.1 The estimator

In summary, we estimate the power spectrum as follows.

#### 7.3.1.1 Constant weighting

When we do not employ the weighting function, first calculate

\[
F(n_r) = N_g(n_r) - \alpha N_s(n_r)
\]  (7.111)

and Fourier transform it. Then, the square of the Fourier transform becomes

\[
\langle |F_{DFT}(n_k)|^2 \rangle = \frac{1}{V^2 w^2 H^2} \sum_{n_q} |W(n_k - n_q)|^2 \tilde{P}(n_q) + H^3 \frac{1 + \alpha}{V} \sum_{n_r} \frac{W^2(n_r)}{n_r}
\]

\[
= \tilde{P}(n_k) + \frac{1 + \alpha}{n}
\]  (7.112)

where the window function is given by

\[
W(n_r) = \frac{\tilde{N}(n_r)}{H^3} \simeq \frac{\alpha N_s(n_r)}{H^3}
\]  (7.113)

and its Fourier transform is

\[
W(n_k) = H^3 \sum_{n_r} W(n_r)e^{-i2\pi n_r n_k/N}
\]  (7.114)
As we distribute particle numbers to regular grid points, $F_{DFT}(n_k)$ has to be deconvolved by the method we described in 7.1.3.

The normalization factor $W^2$ is given by

$$W^2 = \frac{1}{H^3} \sum_{n_r} \tilde{N}^2(n_r) \simeq \frac{\alpha^2}{H^3} \sum_{n_r} N_s^2(n_r). \quad (7.115)$$

Note that we can approximate the average number of galaxies $\bar{N}(n_r)$ by

$$\bar{N}(n_r) \equiv \int d^3r \bar{n}(r) \simeq \alpha N_s(n_r). \quad (7.116)$$

With this approximation, and $\bar{n}(n_r) \simeq \alpha N_s(n_r)/H^3$ the shot noise term may be approximated as

$$H^3 \left(1 + \frac{\alpha}{W^2} \sum_{n_r} W^2(n_r) \right) \bar{n}(n_r) \simeq H^3 \left(1 + \frac{\alpha}{\alpha} \right) \sum_{n_r} N_s(n_r) \sum_{n_r} N_s^2(n_r). \quad (7.117)$$

### 7.3.1.2 FKP optimal weighting

When estimating power spectrum with an optimal weighting function

$$w(n_r) = \frac{1}{1 + \bar{n}(r) P(k)} \simeq \frac{1}{1 + \alpha N_s(r) P(k)/H^3} = \frac{H^3}{H^3 + \alpha N_s(r) P(k)}, \quad (7.118)$$

first calculate

$$F(n_r) = w(n_r) \left[ N_s(n_r) - \alpha N_s(n_r) \right] \quad (7.119)$$

and Fourier transform it. Then, the square of the Fourier transform becomes

$$\frac{1}{W^2} \left\langle |F_{DFT}(n_k)|^2 \right\rangle = \frac{1}{W^2 V} \sum_{n_k} \left| W(n_k - n_k) P(n_k) \right|^2 \left(1 + \alpha \right) \sum_{n_r} \frac{W^2(n_r)}{\bar{n}(n_r)}, \quad (7.120)$$

where the window function is given by

$$W(n_r) = \frac{w(n_r) \tilde{N}(n_r)}{H^3} \simeq \frac{\alpha w(n_r) N_s(n_r)}{H^3} = \frac{\alpha N_s(n_r)}{H^3 + \alpha N_s(r) P(k)}, \quad (7.121)$$

and its Fourier transform is

$$W(n_k) = H^3 \sum_{n_r} W(n_r) e^{-i2\pi n_k \cdot n_r/N}. \quad (7.122)$$

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Then, according to the number distribution scheme, $F_{DFT}(n_k)$ has to be deconvolved by the method we described in 7.1.3. The normalization factor $W^2$ is

$$W^2 = \frac{1}{H^3} \sum_{n_r} w^2(n_r) \bar{N}^2(n_r) \simeq H^3 \sum_{n_r} \left[ \frac{\alpha N_s(n_r)}{H^3 + \alpha N_s(n_r) P(k)} \right]^2,$$  \hspace{1cm} (7.123)

and the shot noise term may be approximated as

$$H^3 \frac{1 + \alpha}{W^2} \sum_{n_r} \frac{W^2(n_r)}{\bar{n}(n_r)} \simeq H^3 \left( \frac{1 + \alpha}{\alpha} \right) \left( \sum_{n_r} \frac{N_s(n_r)}{(H^3 + \alpha N_s(n_r) P(k))^2} \right) \left( \sum_{n_r} \frac{N^2_s(n_r)}{(H^3 + \alpha N_s(n_r) P(k))^2} \right)^{-1}.$$  \hspace{1cm} (7.124)