Homework #8: Due Oct. 24, 2008

1. Find the interval of convergence for each of the following power series.

(a) \( \sum_{n=1}^{\infty} \frac{x^n}{2^n \sqrt{n}} \)

(b) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2} \)

2. Determine the radius of convergence, \( R \), of the power series \( \sum_{n=0}^{\infty} \sin(n) x^n \).

Hint: Note that the ratio and root tests fail to produce the radius of convergence of this power series, so you will have to make use of the other tests for convergence. Use the fact that \( \lim_{n \to \infty} \sin(n) \) does not exist (see Example 5.4C on page 70). Make sure to justify why the series converges for \( |x| < R \) and diverges for \( |x| > R \).

3. Consider the power series expansions

\[
c(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\]

and

\[
s(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.
\]

Find the radius of convergence for each of the above power series and prove that

\( c(x)^2 + s(x)^2 = 1 \)

for all \( x \) by showing that both sides of the identity have the same power series representation.

Hint: The following identity may come in handy.

\[
\sum_{k=0}^{n} \frac{1}{(2k)!(2n-2k)!} = \sum_{k=1}^{n} \frac{1}{(2k-1)!(2n-2k+1)!} = \frac{2^{2n-1}}{(2n)!}
\]

4. Suppose that \( f(x) \) is periodic and increasing. Show that \( f(x) \) must be a constant.