1. Suppose that $f(x)$ is continuous for all $x$ and $f(x) = 0$ if $x$ is rational. Prove that $f(x) = 0$ for all $x$.

   Hint: Assume that there exists an irrational number $a$ such that $f(a) \neq 0$ and show how this would lead to a contradiction.

2. Suppose that $f(x)$ has a jump discontinuity at $a$. Show that $(x-a)f(x)$ is continuous at $a$.

3. Consider the function $f(x)$ defined on $(0,1)$ as follows:

   $f(x) = \begin{cases} 1/2^n & \text{if } x = b/2^n \text{ for positive integers } b,n \text{ and } b \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$

   Show that $f(x)$ is not continuous at $a$ if $a = b/2^n$ for some odd integer $b$ and positive integer $n$.

4. Show that $f(x)$ (as defined in the previous problem) is continuous at all other points in $(0,1)$.

   Hint: Suppose $a$ cannot be written as $b/2^n$. Let $\epsilon > 0$ be arbitrary and pick an integer $N$ such that $1/2^N < \epsilon$. Consider the set

   $X = \left\{ \frac{1}{2N-1}, \frac{2}{2N-1}, \frac{3}{2N-1}, \frac{4}{2N-1}, \frac{5}{2N-1}, \ldots, \frac{2^{N-1}-1}{2N-1} \right\}$

   and pick $0 < \delta < \min(\{|x-a| : x \in X\})$. Show that $|f(x) - f(a)| < \epsilon$ if $|x-a| < \delta$. In the event that $|b/2^n-a| < \delta$, use contradiction to show that $n \geq N$. 