Homework #1: Due Sept. 5, 2008

1. Using only the definition of the absolute value function and the following facts about inequalities/absolute values

   if \( a \leq b \) and \( b \leq c \) then \( a \leq c \) \hspace{1cm} (1)
   if \( a \leq b \) and \( c \geq 0 \) then \( ac \leq bc \) \hspace{1cm} (2)
   if \( a \leq b \) and \( c < 0 \) then \( ac \geq bc \) \hspace{1cm} (3)
   \( |a| < \epsilon \) is equivalent to \( -\epsilon < a < \epsilon \) \hspace{1cm} (4)
   \( |ab| = |a||b| \) \hspace{1cm} (5)

prove the following statements:

(a) If \( 0 \leq a \leq b \) then \( a^2 \leq b^2 \) (i.e., \( x^2 \) is an increasing function for \( x \geq 0 \)).
(b) If \( a^2 < b^2 \) then \( |a| < |b| \) (i.e., \( \sqrt{x} \) is an increasing function).

Make sure to justify every statement (no matter how trivial you think it is) by referring to one of the above facts. Do not simply square both sides or take the square root of both sides of an inequality, since these are precisely the rules that you are trying to prove! You may use the result from part (a) to prove part (b).

2. Show that \( a_n = 1 \cdot \frac{1}{n^2} \cdot \frac{2}{n^2} \cdot \cdots \cdot \frac{n}{n^2} \) for \( n \geq 1 \) is bounded. Is \( \{a_n\} \) monotonic? Justify your answer.

3. Show that \( a_n = \sqrt{n^2 + n} - n \) for \( n \geq 0 \) is bounded. Is \( \{a_n\} \) monotonic? Justify your answer (without taking derivatives).

4. Use induction to prove the extended triangle inequality:

   \[ |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \]

   for \( n \geq 2 \). State, but do not prove, the base case (i.e., \( n = 2 \) case), as this was already done in class.