The Benjamin–Feir instability and propagation of swell across the Pacific

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Abstract

About 40 years ago, Snodgrass and other oceanographers (1966) tracked ocean swell propagating across the entire Pacific Ocean. At about the same time, several investigators (including Benjamin and Feir) showed that a uniform train of plane waves of finite amplitude on deep water is unstable. Comparing these two results, each of which is highly cited, leads to the following question: in light of this instability, how did the waves tracked by oceanographers travel coherently more than 10,000 km across the Pacific Ocean? A possible explanation is provided in recent work that re-examined the Benjamin–Feir instability in the presence of linear damping. The conclusion was that even small amounts of damping can stabilize the instability before nonlinear effects become important. In addition, the theoretical predictions agree well with results from laboratory experiments. In this paper we re-examine ocean data from 1966 to estimate whether the oceanic damping that was measured could have controlled the Benjamin–Feir instability for the swell that was tracked. We find that for one set of ocean swell, dissipation controls the Benjamin–Feir instability enough to allow coherent wave propagation across the Pacific. For a second set of ocean swell, it does not. For a third set of ocean swell, an integral that the theory predicts to be constant is not constant in the data; it decreases and correspondingly the spectral peak shifts to a lower frequency—this is frequency downshifting. For this case the theory is not an adequate model, so the corresponding Benjamin–Feir analysis can be misleading. Thus, our results from the historical records are inconclusive: we can assert neither that dissipation of ocean swell is always negligible, nor that it is always important. But our results show that dissipation can control the Benjamin–Feir instability for small-amplitude waves and that downshifting occurs in ocean swell with relatively small wave slopes.

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1. Introduction

It has been known for at least 60 years [3] that waves generated by focused wind events can propagate away from the wind’s influence and travel more than 10,000 km over the world’s oceans. Linear theory predicts that these waves are dispersive so that away from the storm the wave spectra become narrow-banded (and are called “swell”). One can use the linear dispersion relation to predict the dominant frequency at any distance away from the storm at a given time. Snodgrass et al. [29] used this idea to track twelve swell systems from different storms as they propagated from near
New Zealand to Alaska. These linear techniques are still useful in tracking waves and making swell source predictions, as done, for example, in Refs. [11,21].

What are the effects of nonlinearity on swell propagation? Phillips [26] pioneered resonant interaction theory for water waves, and there is now a large literature on that subject. Snodgrass et al. [29] considered resonant interactions in their study of swell propagation across the Pacific, based on work by Hasselmann [13–15]. They concluded that resonant interactions are important only near the storm where the wave spectra are broad-banded. Snodgrass et al. [29] found significant energy dissipation near the storm that they attributed to spectral spreading due to these wave–wave interactions. But away from the storm, where the spectra are narrow-banded due to dispersion, they saw negligible energy dissipation. Near the time of this oceanographic experiment, Benjamin and Feir [4] and others ([19,5,35,36,33]) found that nearly monochromatic, gravity-driven, plane waves of finite amplitude are modulationally unstable, i.e. they are unstable to a perturbation of other waves with nearly the same frequency and direction. Zakharov [36] showed that this modulational instability, called the Benjamin–Feir instability for water waves, is predicted by the nonlinear Schrödinger (NLS) equation, which describes the entire evolution of waves in a narrow bandwidth. The question then arises: if ocean swell is unstable to the Benjamin–Feir instability, how were Snodgrass et al. [29] able to track specific swell systems across the Pacific Ocean?

In this paper, we investigate the effects of damping on the observations of Ref. [29] following the work of Segur et al. [27]. In Ref. [27] the authors revisited the Benjamin–Feir instability and included a Rayleigh-type damping in the NLS equation. They found an exact, dissipative solution, and examined its stability to modulational perturbations. Their result is that dissipation stabilizes the Benjamin–Feir instability. They also conducted laboratory experiments on plane waves of finite amplitudes. They compared the measured evolution of the Fourier amplitudes of the carrier wave, the sideband perturbations, and two sets of harmonics of the perturbations (seven Fourier amplitudes in all) with predictions of the non-dissipative NLS equation, the dissipative NLS equation, and a higher-order, non-dissipative NLS equation that allowed for energy to spread among sidebands. They found that the dissipative NLS equation predicted all seven of the measured amplitudes very well, while the non–dissipative NLS and higher-order NLS equations, corrected a posteriori for dissipation, did not agree, even qualitatively, with measurements. This good agreement with the dissipative theory occurred when amplitudes were small to moderate. When amplitudes were large, an integral that is related to the momentum integral (call it $P$) and that is conserved by both the non-dissipative and dissipative models, was not conserved in the experiments. Instead, frequency downshifting was observed. (For a discussion of downshifting, see the appropriate section in the review paper by Dias and Kharif [8].) Thus, Segur et al. [27] concluded that for plane waves of moderate amplitudes, dissipation is required in NLS-type models (whether higher order in band-width or not) for agreement with experiments. They also observed that for waves of large enough amplitude, $P$ was not conserved in the experimental data, so NLS-type models (with or without dissipation) simply do not apply. Nonconservation of $P$ then provides one definition of “large amplitude”. These results were further corroborated by Wu et al. [34], who used numerical computations of a dissipative version of the full Euler equations, utilizing the measured damping rates and initial data from Ref. [27].

So the inclusion of dissipation changes the mathematical prediction of instability to a prediction of stability. This prediction agrees well with laboratory experiments, and the inclusion of dissipation in the PDE is required for even qualitative agreement with experiments. What are the consequences for ocean waves? Snodgrass et al. [29] determined that damping rates for ocean swell are negligible in terms of energy dissipation. Here we consider whether the small damping rates they measured can be important in the propagation of ocean swell through a possible effect on the Benjamin–Feir instability. To this end, we re-examine the appropriate data in Ref. [29] in the framework of the Benjamin–Feir stability analysis using the NLS equation and the dissipative NLS equation with damping rates obtained from the ocean data. Snodgrass et al. [29] tracked several storms; the data for three of these were presented in a form that we could use for an analysis similar to what was done for the laboratory system. In particular, we examine the quantity $P$ that is predicted to be conserved by both the dissipative and non-dissipative theories. If it is conserved in the ocean data, then we compute the growth of the “most unstable” (defined in Section 2) perturbation and determine whether dissipation might have been important in bounding its growth.

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1. (i) below 75 mc/s the attenuation is too low to be measured (<0.05 dB/deg). (ii) At 75 and 80 mc/s the attenuation is of the order of 0.1 dB/deg for the large events, but less than 0.05 dB/deg for the small events and background. [29], p. 473.
We find that (i) in one set of ocean swell $P$ is conserved, and the inclusion of the observed damping rate bounds the growth of the perturbation to about four times its initial amplitude. So, for the waves from this storm, dissipation probably was important in the evolution of ocean swell through its effect on the Benjamin–Feir instability. (ii) In a second set of ocean swell, $P$ is conserved, and the inclusion of the observed damping rate bounds the growth of the perturbation to about 60 times its initial amplitude. For the waves from this storm, dissipation probably did not play a qualitative role on the long-term growth of the Benjamin–Feir instability—nonlinearity likely became important in bounding the perturbation growth before the dissipative bound could come into effect. (iii) In a third set of ocean swell, the quantity $P$ is not conserved and the energy spectra show a corresponding downshifting of the spectral peak. Since the Benjamin–Feir theory requires that $P$ be conserved, the authors of Ref. [27] concluded that for cases such as this, any NLS-type model is too simplistic to describe the evolution of the observed waves, so questions concerning the Benjamin–Feir analysis are moot.

Thus, our major conclusions are that: (1) dissipation can play a non-negligible role in the evolution of ocean swell by slowing and eventually stopping the Benjamin–Feir instability for waves of small amplitude; and (2) downshifting occurs in ocean swell with relatively small wave slopes, so that questions about the Benjamin–Feir instability are not central to understanding the swell evolution. This work is preliminary; nevertheless, the results suggest that further investigation with more complete data sets is worthwhile.

We note that the mechanism to stabilize the wave trains discussed herein is not the only possibility. A different stabilizing mechanism, within the NLS framework, was proposed by Alber in Ref. [2], who allowed for a narrow-banded spectrum of random waves (spectra that satisfy Guassian statistics), and considered the competition between nonlinearity (represented by $\epsilon$) and broad-bandedness (represented by $\sigma$). He found a threshold, $2\epsilon / \sigma = 1$, such that random spectra with parameters below/above the threshold are predicted to be stable/unstable. This work was generalized by Crawford et al. in Ref. [7], using the Zakharov integral equation for random waves in two horizontal directions. By further taking the narrow-banded limit, the authors obtained essentially the same stability criterion, where $\sigma$ corresponded to the bandwidth of the spectra in the main direction of propagation. So, this stability criterion says that if a narrow-banded random spectrum is broad-banded enough, so that $2\epsilon / \sigma < 1$, then the spectrum is in steady-state and exhibits no further spreading. If above 1, then the spectrum continues spreading until the ratio equals 1.

In Ref. [29], the investigators obtained spectra at fixed locations from a sequence of (overlapping) 3-h time series for a few days. For two of the wave systems, they show these individual spectra hourly. Then for each measurement site they present one overall spectrum, which represents the swell system at that location. These overall spectra comprise the peaks of each of the individual (3-h) spectra at the corresponding location, and so include information for time series that occur in a period of about 3 days. Herein, we show the overall spectra: one spectrum for each location representing the waves from one storm. The overall spectra from the three wave systems we consider here have ratios, $2\epsilon / \sigma$, of about (.05, .09, .18), respectively, which are well below the instability threshold of 1. For two of the wave systems, the 3-h time series are available. The values of $2\epsilon / \sigma$ for these spectra are about (.55, .88, unavailable). The third storm for which the 3-h spectra are unavailable had a value of $\epsilon$ that was twice that of the middle storm, so we can guess that its 3-h time series had a ratio larger than 1. This significant difference in the values of the stability ratio indicates that there is necessarily some ambiguity in applying the stability result to oceanographic data.

Separately, the stability criterion in Ref. [2] was tested numerically by Dysthe [10], who integrated the NLS equations and higher-order (in bandwidth) versions given in Refs. [9,31,32]. Their numerical results with the NLS equation show that the stability criterion of Ref. [2] is approximately valid for narrow-banded, random waves propagating in one horizontal direction, but not in two horizontal directions. If directional spreading is allowed, then the spectra continue spreading in the transverse direction. Eventually, the directional spread renders the NLS approximation invalid. Integrations using the higher-order in bandwidth models (in either one or two horizontal directions) also showed spreading regardless of the Alber stability threshold. In these numerical simulations the spectra exhibited downshifting with a decrease in the spectral peak, and in 2-d evolved to a particular shape.

A normalization of the nonlinearity to bandwidth ratio (see for example, Ref. [16]), dubbed the Benjamin–Feir Index (BFI) (see Ref. [28] for a discussion of computing the BFI) provides information on the statistical properties of long-crested, narrow-banded spectra. This idea has been used in experiments [24] to study the probability of the occurrence of rogue waves. Further investigation is required to examine the effects of broad-bandedness on the oceanographic data of [29].
In the following we briefly outline the Benjamin–Feir stability analysis and application to the data presented in Ref. [29] in Section 2, show the results for waves from the three storms in Section 3, and list preliminary conclusions in Section 4.

2. Theoretical considerations

Here we consider the NLS-type models that we use to analyze the data of Snodgrass et al. [29] and describe how we apply them to their data. In Section 2.1 we review the inviscid and dissipative NLS equations and two conserved quantities of these systems that can be monitored in the laboratory and in the oceanographic data. In Section 2.2 we discuss the stability analyses for the uniform amplitude solutions, and in Section 2.3 we discuss how we apply this NLS-framework to the oceanographic data in [29].

2.1. Inviscid and dissipative NLS models

For nearly monochromatic plane waves of small amplitude on deep water, we can represent the vertical displacement of the (one-dimensional) water surface as

\[
\eta(x, t; \epsilon) = \epsilon[\psi(T, X)e^{i\theta} + \psi^*(T, X)e^{-i\theta}] \\
+ \epsilon^2[\psi_1(T, X)e^{i\theta} + \psi_1^*(T, X)e^{-i\theta}] \\
+ \epsilon^2[\psi_2(T, X)e^{2i\theta} + \psi_2^*(T, X)e^{-2i\theta}] + O(\epsilon^3),
\]

where \(\epsilon > 0\) is a formal small parameter; \((\cdot)^*\) indicates complex conjugate; \((x, t)\) are physical (horizontal) space and time; \(\theta = k_0x - \omega(k_0)t\) gives the fast oscillation of a carrier wave with wavenumber \(k_0\) and frequency \(\omega_0 = \omega(k_0)\); \(\omega^2(k) = g/k\) for inviscid waves on deep water under the influence of gravitational acceleration, \(g\); \(T = \epsilon\omega_0(t - x/c_g)\) is an appropriate variable to describe the slow modulation of the wave envelope in a coordinate system moving with speed \(c_g = d\omega / dk\); and \(X = \epsilon^2k_0x\) is a slow spatial variable in which to observe the evolution of the envelope as it propagates down the tank.

Zakharov [36] showed that an NLS equation equivalent to

\[
i\psi_X + \alpha\psi_{TT} + \gamma|\psi|^2\psi = 0
\]

is a model for the slow evolution of the complex envelope, \(\psi\), of a nearly monochromatic wavetrain of plane waves of moderate amplitude on deep water. Here we have written the NLS equation so that the evolution is in space, corresponding to wave propagation down a wavetank or across an ocean. The coefficients, \(\alpha\) and \(\gamma\) depend on the frequency and wavenumber of the underlying carrier wave and are given, for example, in [1]. Laboratory experiments [27] show that the effects of dissipation are well modeled by the generalization

\[
i(\psi_X + \delta\psi) + \alpha\psi_{TT} + \gamma|\psi|^2\psi = 0,
\]

where \(\delta > 0\) is a measured, spatial decay rate. This dissipative version of NLS was also used in [18] and [22] for water waves and in [20,12] and [17] for optics. Following [27], one can also consider evolution in a reference frame in which decay is factored out so that

\[
\psi(T, X) = \mu(T, X)e^{-\delta X},
\]

and (3) can be written in terms of \(\mu\) as

\[
i\mu_X + \alpha\mu_{TT} + \gamma e^{-2\delta X}|\mu|^2\mu = 0.
\]
Both (2) and (5) conserve two integrals of the motion that we monitor in applications to determine the validity of these equations as models of the application. For (2) they are

\[ M_\psi = \frac{1}{T_D} \int_D \int_D |\psi(T, X)|^2 dT \]
\[ P_\psi = \frac{i}{T_D} \int_D \int_D [\psi \psi_T - \psi_T \psi] dT, \]

where \( T_D \) is the (non-dimensional) time interval of the domain. The conserved quantities in the presence of dissipation, as in (5), are

\[ M = M_\psi e^{2\delta X} \]
\[ P = P_\psi e^{2\delta X}. \]

An important conclusion of [27] is that if these quantities are not conserved by the application, then the corresponding evolution equation does not apply. Thus, they used the conservation of \( M \) to determine \( \delta \), the damping rate, in their experiments on deep-water laboratory waves. Using that damping rate, they monitored \( P \) to conclude that when \( P \) is conserved, the NLS equation might apply; when \( P \) is not conserved, the NLS equation does not apply. Correspondingly, they identified “small to moderate” amplitude waves to be those for which \( P \) is conserved and “large” amplitude waves to be those for which \( P \) is not conserved. Herein, we also monitor these conserved quantities in the oceanographic data obtained by Snodgrass et al. in [29].

An exact solution to (3), representing a wavetrain with uniform amplitude, is

\[ \psi_0(X) = Ae^{-\delta X} \exp \left\{ \frac{iy|A|^2(1 - e^{-2\delta X})}{2\delta} \right\}. \]

In the reference frame where decay is factored out, (5), the corresponding exact solution, obtained in [27], is

\[ \mu_0(X) = A \exp \left\{ \frac{iy|A|^2(1 - e^{-2\delta X})}{2\delta} \right\}. \]

In the limit of no dissipation (\( \delta \to 0 \)) these solutions are equivalent and correspond to the classic Stokes [30] solution of a uniform wavetrain with a constant frequency correction due to nonlinear effects.

2.2. Linear stability of the exact solutions

We are interested in the linear stability of the uniform amplitude solution (9). A precise definition of stability is given in [27]. Qualitatively, it states that \( \mu_0 \) is a stable solution of (5) if every solution of (5) that starts close to \( \mu_0 \) at \( X = 0 \) remains close to it for all \( X > 0 \). To this end, we follow [27] and consider perturbations of (9) so that:

\[ \mu(T, X) = \exp \left\{ \frac{iy|A|^2(1 - e^{-2\delta X})}{2\delta} + i \arg(A) \right\} \]
\[ \{ |A| + \zeta u(T, X) + i \zeta v(T, X) + O(\zeta^2) \}, \]

where \( \zeta \) is a formal small parameter. We substitute (10) into (5) and keep terms linear in \( \zeta \) to obtain linear partial differential equations for \( u \) and \( v \). Without loss of generality we seek solutions of the form

\[ u(T, X) = U(X; m)e^{i m T} + U^*(X; m)e^{-i m T}, \]
\[ v(T, X) = V(X; m)e^{i m T} + V^*(X; m)e^{-i m T}, \]

to reduce the linear PDEs to coupled linear ODEs for a single Fourier mode of the envelope:

\[ V'(X) = -(am^2 - 2\gamma e^{-2\delta X}|A|^2)U \]
\[ U'(X) = am^2 V. \]

It follows from (12) that the solution in (9) is linearly unstable if there exists \( m^2 > 0 \), such that

\[ am^2(am^2 - 2\gamma e^{-2\delta X}|A|^2) < 0. \]
We note that both $\alpha < 0$ and $\gamma < 0$ for deep-water gravity waves. In the absence of dissipation ($\delta = 0$), there is a set of unstable modes, but with dissipation ($\delta > 0$), each of these modes becomes stabilized with increasing $X$. In other words, small perturbations can grow by no more than a fixed (multiplicative) factor; this factor is determined by the dissipation in the problem, and it can stop the growth before nonlinearity becomes important.

In the absence of dissipation ($\delta = 0$), the largest (spatial) growth rate is

$$\Omega_{\text{max}} = |\gamma||A|^2,$$

(14)

and corresponds to modes with perturbation frequencies

$$m_{\text{max}} = \pm (|\gamma||A|^2/\alpha)^{1/2}.$$  

(15)

For convenience, we say that $|m_{\text{max}}|$ represents the “most unstable mode” because the designation is correct for $\delta = 0$. But we note that with dissipation ($\delta > 0$) the “most unstable” perturbation frequency changes during propagation, and that every mode becomes stabilized in a finite distance. We use (15) and (14) to choose the perturbations for wave spectra obtained in the Pacific Ocean by Snodgrass et al. [29] as discussed in Section 2.3.

2.3. Application to the oceanographic data

To examine the effects of dissipation on the Benjamin–Feir instability as applied to oceanographic data, we follow the procedure set down in [27] for laboratory data, making modifications as necessary and as spelled out herein. However, we make two important distinctions between that work and the present work. (i) The present work is preliminary. It is our first attempt to apply the ideas in [27] to oceanographic data. Segur et al. [27] had all of the information required for testing quantitatively the theories under consideration. Here, we do not have all of the information required, such as the phases of the carrier and sideband waves. We hope to obtain more complete data sets in future work. (ii) The objective in [27] was to test quantitatively two competing theories, the non-dissipative and the dissipative analyses for the Benjamin–Feir instability. The purpose of the present work is necessarily different. Since we do not have phase information, the most we can determine is how much the sidebands could have grown according to the dissipative theory. So here, we are assuming the dissipative theory is correct (since it was tested in the laboratory), and are using it to determine how much dissipation could have affected the measured ocean swell through its effect on the Benjamin–Feir instability.

We begin by expanding the wave envelope function as

$$\mu(T, X) = \sum_{n=-\infty}^{\infty} a_n(X)e^{inbT} (\delta \geq 0),$$

(16)

so that the only frequency differences generated by nonlinear interactions are integer multiples of $b$, the chosen perturbation frequency. Then (5) reduces to an infinite set of complex ODEs, one for every integer $n$:

$$ia_n - \alpha(nb)^2a_n + \gamma e^{-2bX}\sum_{m=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} a_m a_p a_{m+p-n} = 0.$$  

(17)

The quantities predicted to be conserved can be evaluated from the data by

$$M = e^{2bX}\sum_{n=-\infty}^{\infty}|a_n(X)|^2 = \text{constant},$$

(18)

$$P = 2e^{2bX}\sum_{n=-\infty}^{\infty}n|a_n(X)|^2 = \text{constant}.$$  

In this representation, the carrier wave corresponds to the $n = 0$ mode. We are considering a situation in which the carrier wave is perturbed by primary upper and lower sidebands, corresponding to the $n = \pm 1$ modes. The modes with $|n| > 1$ are harmonics of the primary sidebands, and are forced by nonlinear interactions among modes. Accordingly, there is a natural ordering of the sidebands, which we define to be

$$a_0(X) = O(\sqrt{M}); a_1(X) << \sqrt{M}; a_{-1}(X) = O(a_1).$$

(19)
and
\[
a_n(X) = O\left(\left(\frac{a_1}{\sqrt{M}}\right)^{|n|}\right), \text{ for } |n| > 1.
\]
(20)

To determine the evolution of the sidebands, we use this ordering in (17) to obtain a reduced set of coupled ODEs for the sidebands and their harmonics. Here we are interested in the linearized growth of the primary perturbations, so we consider only
\[
\begin{align*}
    ia_{-1}'(X) - ab^2a_{-1} + ye^{-2\delta X}(2a_{-1}|a_0|^2 + a_0^2a_{-1}^*) &= 0(a_1^3) \\
    ia_1'(X) - ab^2a_1 + ye^{-2\delta X}(2a_1|a_0|^2 + a_0^2a_{-1}^*) &= 0(a_1^3).
\end{align*}
\]
(21)

Then (again following [27]) we define
\[
\begin{align*}
    a_{-1}(X) &= \frac{1}{2}\{u(X) + iv(X) + i[U(X) + iV(X)]e^{i\phi(X)/2 + i\arg A} \\
    a_1(X) &= \frac{1}{2}\{u(X) + iv(X) - i[U(X) + iV(X)]e^{i\phi(X)/2 + i\arg A},
\end{align*}
\]
(22)

where \(\phi(X) = (\gamma/|A|^2/\delta)(1 - e^{-2\delta X})\), and \(u, v\) and \(U, V\) each satisfy (12). In Section 3 we discuss numerical solutions of (12) in order to examine the effect of measured dissipation on the evolution of the oceanographic wave systems measured by Snodgrass et al. [29] using the following procedure:

1. Spectra at measurement sites on a great circle route across the Pacific are available in [29] for some of the wave systems they observed. These spectra were corrected by Snodgrass et al. [29] for geometric spreading, water depth and island shadowing, and we use their corrected data. (The units of the spectra are given in dB and are relative to 1 cm²/(m·s).) Data from three of their wave systems provide the necessary information for the analysis described herein. We digitized the spectra from three figures in their paper by hand and converted the resulting data to the form used in the laboratory experiments by Segur et al. [27] using
\[
S(f) = E_0 f^P(f)/10\mathrm{dB}.
\]
(23)

In (23), \(P(f)\) (in units of dB) is the value digitized from the figures in [29]; \(E_0 = 1\mathrm{cm}^2/(\mathrm{m} \cdot \mathrm{s})\); \(S(f)\), in units of \(\mathrm{cm}^2\), is the Fourier amplitude squared; and \(f\) is frequency in units of \(\mathrm{m} \cdot \mathrm{s}\). Snodgrass et al. [29] provided no phase information for their data.

2. Table 1 of [29] provides two measures (which are typically quite close) of the storm sources’ locations and the distances from the storm sources to the measurement sites in units of degrees. We average these sets of two values. So \(x = 0\) (\(x\) is dimensional) corresponds to the location of the storm source and is different for each storm. Then \(x_n\) is the distance to the \(n\) th measurement site and is different for each storm. The results shown in Section 3 are given in dimensional quantities so that \(x\) is a physical distance in units of degrees, where \(1\mathrm{deg} = 69\mathrm{mile}\).

3. Once we have \(S(f)\) at different \(x\)-locations, we use (18) to compute the values of \(M\) and \(P\) as functions of \(x\). We take the log of the results for \(M\) at the different measurement sites and fit a line. The slope of the line is the “measured” value of \(2\delta = 2e^2k_0\delta\), twice the dimensional, exponential damping rate (with units of \(1/\mathrm{deg}\)). The graphs of \(M\) shown in Section 3 are in the reference frame that factors out this decay rate so that the horizontal lines shown as reference correspond to an exponentially decaying curve with decay rate \(2\delta\) in the ocean reference frame. We then use this damping rate to graph \(P\) in the reference frame with decay factored out. If \(P\) stays “close” to a horizontal line in this reference frame then it is judged to be conserved, as required by the theory.

4. To apply the theory, we need \(\omega_0\), the frequency of the carrier wave, and its normalized amplitude, \(A\), as in (9). (Recall that \(A\) has dimensions of length.) We pick the peak of the amplitude spectrum for a given storm system at the first measurement site and call its amplitude \(a_0 = 2A\) and its corresponding frequency, the \(n = 0\) carrier wave frequency. Again, there is no phase information available, so we let \(\arg a_0 = \arg A = 0\).

5. In their laboratory experiments, Segur et al. [27] imposed a perturbation on the carrier wave. So they knew the perturbation frequency and its initial amplitude and phase. Here, there is no dominant perturbation, so we examine the growth of the sidebands that correspond to the perturbation mode with the largest non-dissipative growth rate,
given by (14). Then the perturbation frequencies are \(\omega_0(1 \pm b)\), where \(b\) is from (16), and we choose it to correspond to the most unstable mode given in (15), so that \(b = 1\) \(m_{\text{max}}\) \(l\).

(6) The experiments in [27] showed that initial phase information is important in predicting the initial growth of sidebands. Consider (12) for zero dissipation \((\delta = 0)\). The equations are of second order, so unstable solutions could start out with exponential decay, exponential growth or some combination, depending on the initial phase. Here we have no phase information. So, we choose initial data that result in purely exponential growth at the maximum rate as given by the inviscid case. We choose \(u(0) = 1\) and \(U(0) = 1\); the bounds that we obtain on total growth are independent of this choice. Then

\[
v(0) = u(0)[(-\hat{\alpha}\hat{b}^2 + 2\hat{\gamma}|A|^2)/\hat{\alpha}\hat{b}^2]^{1/2}
\]

\[
V(0) = U(0)[(-\hat{\alpha}\hat{b}^2 + 2\hat{\gamma}|A|^2)/\hat{\alpha}\hat{b}^2]^{1/2}.
\]

where \(\hat{b} = \omega_0b\) is the perturbation frequency \((\text{in rad/s})\) with the maximum growth rate; \(\hat{\alpha} = \alpha k_0/\omega_0^2; \hat{\gamma} = \gamma k_0; \alpha = 1\) and \(\gamma = -4k_0^2\) for gravity waves on deep water; and \(\{k_0, \omega_0\}\) are the wavenumber and frequency of the carrier wave. In this way, we obtain upper bounds on the maximal growth factors of perturbed sidebands, before nonlinear interactions among sidebands can affect their growth. Without initial phase information, this is the best we can do.

(7) We use the damping rate from Step 3 and the initial data from Step 6 to integrate numerically (using Mathematica) the ODEs, (12), for \(\{u, v\}\) and \(\{U, V\}\). In dimensional form then we compute:

\[
v'(x) = -(\hat{\alpha}\hat{b}^2 - 2\hat{\gamma}e^{-2\hat{\gamma}k_0}|A|^2)u,
\]

\[
u'(x) = \hat{\alpha}\hat{b}^2v,
\]

\[
V'(x) = -(\hat{\alpha}\hat{b}^2 - 2\hat{\gamma}e^{-2\hat{\gamma}k_0}|A|^2)U,
\]

\[
U'(x) = \hat{\alpha}\hat{b}^2V.
\]

(8) We repeat Step 7 using \(\delta = 0\) to determine the inviscid results.

(9) We use (21) to put the results for \(\{u, v, U, V\}\) obtained from Step 7 together to obtain the maximally growing amplitudes of the left and right-hand sidebands for both the dissipative and non-dissipative results. We note that the reference frame we use in comparing the dissipative and non-dissipative results has the damping rate factored out so that the inviscid growth rate has the measured decay rate subtracted out from it.

3. Results

The question we ask is: can dissipation play a role in the evolution of ocean swell through its effect on the Benjamin–Feir instability? The answer in [27] from the theory is that dissipation slows and eventually stops the growth of small perturbations, and it can bound their total growth before nonlinear effects set in. The answer in [27] from comparisons of laboratory results with the theory is that (i) dissipation can bound the growth of small enough initial perturbations before nonlinearity becomes important and is essential in describing observations; (ii) for large amplitude waves, \(P\) is not conserved, so corresponding questions of stability are moot.

Here we look at the results from waves from three storms. Table 1 lists the locations of the measurement sites relative to the storms’ locations. (See Fig. 1 in Snodgrass et al. [29] for the precise location of each storm.) Table 2 lists values of parameters we obtained from the spectra in [29] and from associated theoretical considerations as outlined in Section 2.3. Frequencies are listed in units of mc/s; a frequency of \(f = 50\) mc/s corresponds to a 20 s period.

Table 1

<table>
<thead>
<tr>
<th>Storm</th>
<th>Cape Palliser (deg)</th>
<th>Tutuila (deg)</th>
<th>Palmyra (deg)</th>
<th>Honolulu (deg)</th>
<th>Yakutat (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.2 July</td>
<td>no obs</td>
<td>52.2</td>
<td>75.6</td>
<td>88.3</td>
<td>128.8</td>
</tr>
<tr>
<td>1.9 August</td>
<td>no obs</td>
<td>21.8</td>
<td>43.5</td>
<td>58.5</td>
<td>99.3</td>
</tr>
<tr>
<td>13.7 August</td>
<td>0</td>
<td>33.7</td>
<td>56.8</td>
<td>71.8</td>
<td>112.5</td>
</tr>
</tbody>
</table>
note that the amplitudes of the waves considered here are representative of the amplitudes of all the waves discussed in [29]. The damping rates that we measure are comparable to those reported by Snodgrass et al. [29], except for the waves from the 1.9 August storm, for which the rates we compute are about twice as big as the values quoted in [29]. We note that Snodgrass et al. [29] reported damping rates for spectral components with particular frequencies. For example, in their abstract, they cite damping rates that are below 0.02 dB/deg for frequencies less than 70 mc/s and a damping rate of 0.15 dB/deg for a frequency of 80 mc/s. (Also, see footnote 1 in our Section 1.) These damping rates correspond to .002/deg and .017/deg in the present units. However, our damping rates result from integrating over all frequencies in the spectra. So, we are looking at the decay of the integral $M$, rather than the decay of the energy in particular spectral components. We also note that the damping rates we deduce from the oceanographic data are less than the values predicted by a model that treats the surface as a boundary-layer that can oscillate vertically but cannot stretch horizontally, i.e. the fully-contaminated-surface model (see for example, [23]). For example, for the highest carrier wave frequency considered here, 65 mc/s, the predicted decay rate from the fully-contaminated-surface model is 0.036/deg.

### 3.1. The July 23.2 storm

Fig. 1a shows the spectra at the four measurement sites for the 23.2 July storm. This figure was obtained by digitizing the data in Fig. 26 of [29] as described in Step 1 in Section 2.3. The $M$ integral is shown in Fig. 1b and obtained from the data in Fig. 1a as described in Step 3. It does not decay monotonically for the waves in this storm—the energies associated with each frequency at the second measurement site (Palmyra) are less than at the third site (Honolulu). We do not know if this loss/gain is scatter in the data or a fundamental issue. In general there is significant scatter in the $M$ and $P$ integrals, but no consistent trend.

To examine the growth of the perturbations, we follow the procedures outlined in Steps 4–9; the most unstable frequency is listed in Table 2. Fig. 2 shows that the dissipation, with no nonlinearity, bounds the growth of the perturbations to a factor of about 4. This bound is reached at about 45 deg from the storm location. The inviscid theory predicts that the perturbations grow by a factor of 4 by about 20 deg from the storm location and that the perturbation continues growing until nonlinearity becomes important. So, the bound on growth due to dissipation could have been important in the subsequent evolution of this wave system.

<table>
<thead>
<tr>
<th>Storm</th>
<th>$\delta$ (1/deg)</th>
<th>$\frac{1000}{2\pi} w_0$ (mc/s)</th>
<th>$2\lambda A_1$ (cm)</th>
<th>$2\lambda A_1 k_0$</th>
<th>$\frac{1000}{2\pi} \hat{\delta}$ (mc/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.2 July</td>
<td>0.013</td>
<td>56.0</td>
<td>56.3</td>
<td>0.007</td>
<td>0.4</td>
</tr>
<tr>
<td>1.9 August</td>
<td>0.025</td>
<td>65.0</td>
<td>84.3</td>
<td>0.014</td>
<td>1.0</td>
</tr>
<tr>
<td>13.7 August</td>
<td>0.014</td>
<td>63.0</td>
<td>181.7</td>
<td>0.029</td>
<td>1.8</td>
</tr>
</tbody>
</table>
Fig. 2. The growth of perturbations normalized by their initial amplitudes for the July 23.2 waves. Solid curves are predictions from the dissipative theory ($\delta = 0.013/\text{deg}$); dashed curves are from the inviscid theory ($\delta = 0$). Computations for (a) the minus sideband and (b) the plus sideband.

Fig. 3. The (a) spectra at Tutuila (circles), Palmyra (squares), Honolulu (diamonds), and Yakutat (triangles) (obtained from [29]); (b) the $M$ integral normalized by the value of $M$ at Tutuila; and (c) the $P$ integral normalized by the value of $P$ at Tutuila, for the August 1.9 waves. The reference frames in (b) and (c) have dissipation factored out so that the horizontal lines there correspond to an exponential curve with a decay rate, $2\delta$, given in Table 2.

3.2. The August 1.9 storm

Fig. 3a shows the spectra at the four measurement sites for the 1.9 August storm. This figure was obtained by digitizing the data in Fig. 21 of [29]. For this storm, the $M$ integral, shown in Fig. 3b, does decay monotonically, with less scatter than in the 23 July storm. The $P$ integral shown in Fig. 3c behaves similarly.

The most unstable perturbation for the waves from this storm is listed in Table 2. Fig. 4 shows that the dissipation bounds the growth of the perturbations to a factor of about 60. With such a large bound, it is unlikely that dissipation played an important role in the overall growth of the Benjamin–Feir instability. This calculation only provides upper bounds on this growth; without initial phase information we have no way of obtaining more accurate estimates of the actual growth. Note that the measure of nonlinearity for this set of waves is $21A1k_0 = 0.014$, about twice that of the 23.2 July waves.

Fig. 4. The growth of perturbations normalized by their initial amplitudes for the August 1.9 waves. Solid curves are predictions from the dissipative theory ($\delta = 0.025/\text{deg}$); dashed curves are from the inviscid theory ($\delta = 0$). Computations for (a) the minus sideband and (b) the plus sideband.
3.3. The August 13.7 storm

Fig. 5a shows the spectra at the four measurement sites for the 13.7 August storm. This figure was obtained by digitizing the data in Fig. 30 of [29]. For this storm, the $M$ integral, shown in Fig. 5b, decays monotonically, with slight scatter. However, the $P$ integral shown in Fig. 5c shows a monotonically decreasing trend. In fact, $P$ switches from positive to negative, which is impossible for a function that is decaying exponentially. So, we see that $P$ is not behaving as predicted by the nonlinear Schrödinger models, (2) and (5). This monotonic decrease is consistent with “frequency downshifting”, when a lower sideband eventually dominates the carrier wave in the spectrum. Indeed, downshifting is visible in Fig. 5a where the peak of the spectrum at the third measurement site (at Palmyra) is lower than the frequency of the carrier wave, which is determined at the first measurement site (at Tutuila). Possible explanations for the phenomenon of downshifting are discussed in [8].

Because the quantity, $P$, is not conserved in the waves from this storm, we consider these waves to lie outside the range of validity of either the dissipative or the non-dissipative NLS theories. As a result, questions about the Benjamin–Feir instability become moot. Here we note that the measure of nonlinearity is $21A1k_0 = 0.029$, about four times that of the 23.2 July wave system, or a sixteen-fold increase in energy.

4. Summary

In this paper we consider whether or not dissipation can play a role in the evolution of ocean swell by having an effect on the Benjamin–Feir instability. We examine wave data from swell that propagated away from three storms in the Pacific as reported by Snodgrass et al. [29] using the theoretical framework of dissipative and non-dissipative NLS equations, their exact solutions, and stability analyses of modulational perturbations of these solutions. The results are as follows:

1. In one set of ocean swell, discussed in Section 3.1, dissipation could have played a role in the evolution of ocean swell through the Benjamin–Feir instability. In particular, dissipation provides a bound to perturbation growth that could preclude nonlinear effects.

2. In another set of ocean swell, discussed in Section 3.2, we find that dissipation provides a bound on perturbation growth. However, the dissipative bound is so large that nonlinearity would probably have been important in bounding perturbation growth before dissipation could have had an effect. Nevertheless, we note that in laboratory experiments the initial phases were important in determining the magnitude of the dissipative bounds as well as the initial evolution of the perturbations. Here we do not have phase information and so choose phases that provide the fastest growth and maximum possible dissipative bounds of the perturbations. Another possible stabilizing mechanism for these waves, which assumes random phases, is the ratio of broad-bandedness to nonlinearity as discussed by Abler [2] and Crawford [7].

3. In a third set of ocean swell, discussed in Section 3.3, the quantity $P$, which is predicted by both the dissipative and non-dissipative theories to be conserved, is not conserved in the data. Instead, it decreases and changes sign. This result corresponds with the observation that the spectral peak shifts from the carrier wave frequency to a lower
sideband. This phenomenon of downshifting in ocean swell (i.e. in the absence of forcing by wind) indicates that both the dissipative and the non-dissipative NLS-type models are inadequate. Segur et al. [27] used this result as a diagnostic that delineates small-to-moderate amplitude waves from large-amplitude waves, with large amplitude waves exhibiting downshifting and a decrease in $P$. Surprisingly, in the ocean swell, this large-amplitude result occurred for a wave slope of $2\Delta A/k_0 = 0.029$, which is far below the wave slope required for breaking. To our knowledge, this is the first documented observation of frequency downshifting in ocean swell.

This preliminary work suggests that wave dissipation can have an effect on the evolution of ocean swell, by slowing and eventually stopping the Benjamin–Feir instability for waves of small amplitude. It also shows how initial phase information can be important. For the waves from July 23.2, the initial phases did not matter; regardless of phase, dissipation bounds perturbation growth by a factor of 4. For the August 1.9 waves, initial phases could have mattered. Dissipation bounds the growth of sidebands by a factor of about 60, but the initial phases of the perturbations might have made their actual growth much smaller, even before nonlinear interactions among sidebands further controlled their growth. For the August 13.7 waves, initial phases did not matter; downshifting occurred so the stability analyses, with or without dissipation, do not apply. The work also suggests that frequency downshifting might occur more commonly than has been acknowledged during the evolution of ocean swell of larger amplitude. Most or all of the currently used theoretical models of nonlinear wave interactions on deep water are inadequate to describe accurately frequency downshifting of ocean swell.

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References