Stable deep-water waves propagating in one and two dimensions

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This work is dedicated to the memory of Joseph Hammack.

Deep-water, narrow-banded wavetrains of uniform amplitude propagating in one horizontal dimension were shown to be unstable to small perturbations with nearly the same frequency and direction about 40 years ago. More recently, interactions of two narrow-banded wavetrains of uniform amplitude propagating in arbitrary directions were shown to be similarly unstable. In both cases, the instabilities have been described by either a single nonlinear Schrödinger (NLS) equation or by two coupled NLS equations. However, the inclusion in these equations of any amount of damping (of a particular type) stabilizes the instabilities. Experiments show that the evolution of perturbed wavetrains in both cases is more accurately described by the NLS models that include damping when the amplitudes are small or moderate. When the amplitude of either the underlying waves or the perturbations is large, neither the undamped NLS model nor the damped NLS model accurately predicts the observed behaviour, which includes frequency downshifting.

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1 Introduction

In a landmark paper, Benjamin & Feir [1] showed that a uniform train of plane waves of moderate amplitude in deep water without dissipation is unstable to a small perturbation of other waves travelling in the same direction with nearly the same frequency. This modulational instability is a finite-amplitude effect, in the sense that the unperturbed wave train (the carrier wave) must have finite amplitude, and the growth rate of the instability is proportional to the square of that amplitude, at least for small amplitudes. More recently, several authors, including [2, 5–7], showed that deep-water waves with two-dimensional surface patterns (so the velocity fields are three-dimensional), resulting from the interaction of two uniform-amplitude wavetrains are similarly unstable. These instability results occur in the undamped case only. If damping effects are included in the model equations, then these instabilities are stabilized as shown by [3, 6, 9]. Here we summarize comparisons of the undamped and damped nonlinear Schrödinger (NLS) models with experiments and verify that any amount of damping, of the right type, stabilizes the instability when the wave amplitudes are small or moderate. Our experiments (not shown here) also show that the question of modulational stability is not central for large amplitude patterns, for which downshifting occurs so that both the undamped and the damped NLS-type models are inadequate.

2 Stabilizing the instability

Zakharov [10] derived the (undamped) nonlinear Schrödinger equation to describe approximately the slow evolution of the complex envelope, ψ, of a nearly monochromatic train of plane waves of moderate amplitude in deep water. Other authors included the effects of damping so that an NLS equation in two spatial dimensions that includes linear damping is

\[ i(\frac{\partial}{\partial t} - \delta)\psi + \alpha \partial_x^2 \psi + \beta \partial_y^2 \psi + \gamma |\psi|^2 \psi = 0, \tag{1} \]

where \( \{\alpha, \beta, \gamma, \delta > 0\} \) are real-valued constants. Segur et al [3] showed that in contrast to the undamped, instability result, the uniform wavetrain solution of this damped NLS equation (1) is stable to long-wave perturbations. Their experiments confirmed this result for waves with moderate amplitudes. An example is given in Figure 1, which shows results from an experiment on a (3.33 Hz) wavetrain (carrier wave), with a long-wave perturbation imposed, propagating in one horizontal dimension. The figure shows a time series measured at a fixed location and its corresponding Fourier Transform. The imposed long-wave perturbation is apparent in the time series as a modulated envelope and in the Fourier Transform as sideband peaks on either side of the carrier wave. The carrier wave is predicted to be (i) unstable by the undamped (\( \delta = 0 \)) version of (1), (ii) stable by the damped (\( \delta > 0 \)) version, and (iii) unstable by an undamped version that also includes higher-order terms. These predictions are depicted in Figures 1(c) and (d), which show results of numerical integrations of (i), (ii), and (iii). All of the numerics used measured amplitudes and phases for initial data (obtained from Figure 1(a) and (b)). The damping rate used in model (ii) was determined by measurements of the energy decay in the time series. The higher-order NLS used in model (iii)

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These models to describe accurately the evolution of small to moderate amplitude wave patterns. The curves show the Fourier amplitudes of the left- and right-hand sidebands as they evolve a distance X from the initial gage location. Also shown are the measured Fourier amplitudes. Both undamped models (which are indistinguishable in this experiment) predict unbounded growth within the length of the wavetank. However, the perturbation did not grow; the wavetrain remained stable, in agreement with the damped-NLS equation, which predicts stabilization.

A similar result holds for the interaction of wavetrains. It was shown in [2, 5–7, 9], that deep-water waves with two-dimensional surface patterns (so the velocity fields are three-dimensional), resulting from the interaction of two uniform-amplitude wavetrains are modulationally unstable within the context of undamped coupled NLS equations. However, as in the case of a single uniform wavetrain, the uniform pattern solution of coupled NLS equations is stable if the damping rates are non-zero as shown by [6, 9].

Figure 2 shows a photograph from [9] of an experiment using two symmetric wavetrains, forming a pattern of rectangular cells delineated by nodal lines of no surface displacement parallel to the direction of propagation (which is from the top to the bottom of the photo). Gages that measured surface displacement were placed in one of these nodal lines and in the antinodal regions on either side. (See [4] for details of the experimental facility and [9] for a comparison with theory.) In this experiment the pattern was seeded with a long-wave perturbation. Figure 2 also shows predictions of the Fourier amplitudes of the left and right hand sidebands in the antinodal regions due to the perturbation, using measured initial amplitudes and phases in the undamped and the damped versions of coupled NLS equations. The damped version, which predicts stability of the pattern, more accurately predicts the measured evolution of the perturbation.

In conclusion, dissipation stabilizes the modulational instability in single and coupled NLS-type models, and is required in these models to describe accurately the evolution of small to moderate amplitude wave patterns.

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