Problem 1. Consider the following game tree:

1.a) Come up with a “real life” story for it. That is, describe with words a strategic environment that could be modelled by it. Your description should specify payoffs. Feel free to rename the players and actions.

It is not important whether your story and payoffs are plausible or not. Your score depends on whether you justified all the elements of the game: (1) there are two players, (2) player 1 gets to make one decision followed by a decision from player 2 and maybe a final decision by player 1, (3) players are not informed about their opponent’s choices at the moment of making their own, and (4) whichever payoffs you specified. The following is an example of a complete description that you could have used.

Suppose that John is going to have lunch downtown and he spots this amazing parking spot right in front of the restaurant where he wants to eat. However he doesn’t have enough quarters for the parking meter. So he has two alternatives: he can park far away in a free spot and walk (W), or he can park at this amazing but illegal spot (I) and risk getting a ticket. On the other hand we have Kevin, the parking officer, who must choose whether to patrol the area looking for violations (P) or not (NP). Kevin must make this decision without knowing whether John is parking illegally or not. John knows that if he spots Kevin before he gets the ticket he will be able to talk himself out of it. So, while he is eating and without knowing whether Kevin is patrolling or not, Kevin has the option to exit the restaurant (E) to see whether Kevin is around or not, or to continue eating in peace (NE). John has to pay a cost of 4 for walking if he parks far away, a cost of 5 if he interrupts his meal to look for Kevin and a cost of 10 if he gets a ticket. Kevin has to pay a cost of 2 to patrol and he gets a bonus of 5 for every ticket that he issues. Recall that John gets a ticket only if (1) he parks illegally, (2) Kevin is patrolling and (3) he doesn’t interrupt his meal to spot Kevin.

1.b) Write down a payoff matrix for a strategic form game representing the environment you described in (1.a). How many strategies does each player have?
John has four strategies \{(W, E), (W, NE), (I, E), (I, NE)\} and Kevin has two strategies \{P, NP\}. The following payoff matrix corresponds to the associated strategic form game.

\[
\begin{array}{c|cc|cc|cc|cc}
& \text{John} & & & & & & \\
\text{Kevin} & (W, E) & (W, NE) & (I, E) & (I, NE) \\
\hline
P & -2, -4 & -2, -4 & -2, -5 & 3, -10 \\
NP & 0, -4 & 0, -4 & 0, -5 & 0, 0 \\
\end{array}
\]

**Problem 2.** Identify which of the following are valid trees with valid information structures. For those that are not valid you must specify why. For those that are valid please specify how many strategies each player has.

**2.a)**

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**2.a)** It is a valid game tree with a valid information structure. Player 1 has 2 strategies, player 2 has 2 strategies and player 3 has \(2 \times 2 \times 3 = 18\) strategies.
2.b) It is a valid game tree but the information structure is **NOT VALID** because it does not satisfy perfect recall. To see this suppose that player 1 moves left at the first node, and then both players 2 and 3 move right in the subsequent nodes. After the first move by player 1, player 2 is informed that player 1 moved left. However, the second time that he gets to make a move, player 2 doesn’t know whether player 1 moved left or right at the beginning of the game: player 2 has forgotten something that he knew in the past!

2.c) It is **NOT A VALID** game tree because the is a node (the fourth terminal node from left to right) that has two predecessors.

2.d) It is a valid game tree with a valid information structure. Player 1 has $3 \times 2 = 6$ strategies, Player 2 has $2 \times 1 = 2$ strategies and player 3 has 3 strategies.

**Problem 3.** Consider the following strategic environment involving the owner of a firm, a manager and a worker. The owner first decides whether to hire the worker, to refuse hiring the worker or to delegate the hiring decision to the manager. If the owner lets the manager decide, then the manager must choose whether to hire the worker or not. If the worker is hired then he/she chooses whether to work diligently or to shirk. Suppose that the worker does not know whether the hiring decision was made by the manager or by the owner. If the worker is not hired then all three players get a payoff of 0. If the worker is hired and shirks then both the manager and the owner get a payoff of $-1$ and the worker gets a payoff of 1. If the worker is hired by the owner and works diligently, then the owner gets 3 and both the worker and the manager get a payoff of 0. If the worker is hired by the manager and works diligently, then the owner gets 0, the manager gets 1 and the worker gets 2.

3.a) Write down an extensive form game that represents this situation

The situation can be described by the following extensive form game in which O stands for owner, E for the worker (E is for employee) and M for manager, H stands for hire, R for refuse, D for delegate, W for work diligently and S for shirk. The payoffs are written from top to bottom starting with the owner, then the manager and then the worker.
3.b) Write down a strategic form game that represents this situation

The environment can be represented by the following strategic form game in which the owner chooses the row, the manager chooses the column and the worker chooses the matrix. We have underlined the payoffs corresponding to best responses for each player.

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>3,0,0</td>
<td>3,0,0</td>
</tr>
<tr>
<td>R</td>
<td>0,0,0</td>
<td>0,0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,1,2</td>
<td>0,0,0</td>
</tr>
</tbody>
</table>


3.c) Which strategies are rationalizable?

Notice from the underlined payoffs that every strategy is a best response to some other pure strategy. For the owner: H is a best response to (H,W) and (R,W), R is a best response to (S,H) and (S,R) and D is a best response to (S,R). For the manager: H is a best response to (H,W), (H,S), (R,W), (R,S) and (D,W) and R is a best response to (H,W), (H,S), (R,W), (R,S) and (D,S). For the worker: W is a best response to (R,H), (R,R), (D,R) and (D,H) and S is a best response to (R,H), (R,R), (D,R), (H,H) and (H,R). Since all strategies are best responses, none of them are dominated. Therefore all strategies are rationalizable.

3.d) [Bonus] Find all the Nash equilibria in pure strategies. Hint: There are three Nash equilibria in pure strategies and all of them yield the same payoffs

There are three pure strategy Nash equilibria: (R,H,S), (R,R,S) and (D,R,S).

Problem 4. Anna and Bob are working as partners. During a given year they will both provide effort and the firm will generate revenues that depend on the levels of effort provided. They can provide any level of effort in [0,5]. Let A denote the level of effort provided by Anna and B the level of effort provided by Bob. Providing effort is costly, the cost for Anna is \(-\frac{1}{2}A^2\) and the cost for Bob is \(-\frac{1}{2}B^2\). The total revenue of the firm is \(2(A+B+\frac{1}{2}AB)\).

Bob and Anna split the revenue halfway so that the total revenue that each one for them receives is \((A+B+\frac{1}{2}AB)\). Notice that we have a simultaneous move game with players \{Anna,Bob\}, strategy sets \(S_{Anna} = S_{Bob} = [0,5]\) and payoff functions \(u_{Anna}(A,B) = A + B + \frac{1}{2}AB - \frac{1}{2}A^2\) and \(u_{Bob}(A,B) = A + B + \frac{1}{2}AB - \frac{1}{2}B^2\).

4.a) Find an analytic solution for the best response functions and graph them in a clearly labelled figure. Hint: Notice that the problem is very similar to the Cournot competition example covered in class.

Notice that the expected payoff functions are concave and quadratic:

\[
U_{Anna}(A,B) = -\left(\frac{1}{2}\right)A^2 + \left(1 + \frac{1}{2}B\right)A + (\bar{B}) \quad U_{Bob}(\bar{A},B) = -\left(\frac{1}{2}\right)B^2 + \left(1 + \frac{1}{2}\bar{A}\right)B + (\bar{A})
\]
Thus, using our formula to maximize concave-quadratic functions we know that the best response functions are:

\[
BR_{Anna}(\bar{B}) = \frac{1 + \frac{1}{2}B}{2} = 1 + \frac{1}{2}\bar{B} \quad BR_{Bob}(\bar{A}) = \frac{1 + \frac{1}{2}A}{2} = 1 + \frac{1}{2}\bar{A}
\]

4.b) Can Anna rationalize choosing \( A = 4 \)? How about \( A = 2.5 \) or \( A = 1.5 \)? Justify your answer in detail.

Neither \( A = 4 \) nor \( A = 2.5 \) nor \( A = 1.5 \) are rationalizable. Recall that rationalizability is equivalent to rationality and common knowledge of rationality:

- Since Anna is rational, she cannot justify choosing \( A = 4 \) because her best response function only takes values between 1 and 3.5, and thus 4 is not a best response to anything.
- Since Bob knows that Anna is rational, he knows that she will choose a strategy between 1 and 3.5. Being rational, this implies that Bob will choose a best response between \( BR_{Bob}(1) = 1.5 \) and \( BR_{Bob}(3.5) = 2.75 \).
- Since Anna knows that Bob is rational and that Bob knows that Anna is rational, Anna knows that Bob will choose a strategy between 1.5 and 2.75. Being rational this implies that her choice must be between \( BR_{Anna}(1.5) = 1.75 \) and \( BR_{Anna}(2.75) = 2.375 \) and thus she cannot rationalize choosing either \( A = 1.5 \) nor \( A = 2.5 \).

We can also show that \( A = 1.5 \) and \( A = 2.5 \) are not rationalizable using a different argument, shown in the following figure. Recall that a strategy is rationalizable if and only if it can be justified by a complete argument. Notice that \( A \) is a best response to \( \bar{B} \) if and only if:

\[
A = 1 + \frac{1}{2}\bar{B} \iff \bar{B} = 2A - 2
\]

and, by a similar argument, \( B \) is a best response to \( \bar{A} \) if and only if \( \bar{A} = 2B - B \). This implies that:
- $A = 1.5$ is only a best response to $\bar{B} = 1$ and $B = 1$ is only a best response to $\bar{A} = 5$. Since $A = 5$ is never a best response, this implies that $A = 1.5$ cannot be justified by a complete argument.
- $A = 2.5$ is only a best response to $\bar{B} = 3$ and $B = 3$ is only a best response to $\bar{A} = 4$. Since $A = 4$ is never a best response, this implies that $A = 2.5$ cannot be justified by a complete argument.

**Problem 5.** Find the set of rationalizable strategies for the following game:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>5, 4</td>
<td>4, 4</td>
<td>4, 5</td>
<td>12, 2</td>
</tr>
<tr>
<td>x</td>
<td>3, 7</td>
<td>8, 7</td>
<td>5, 8</td>
<td>10, 6</td>
</tr>
<tr>
<td>y</td>
<td>2, 10</td>
<td>7, 6</td>
<td>4, 6</td>
<td>9, 5</td>
</tr>
<tr>
<td>z</td>
<td>4, 4</td>
<td>5, 9</td>
<td>4, 10</td>
<td>10, 9</td>
</tr>
</tbody>
</table>

The only rationalizable strategy for player 1 is $x$ and the only rationalizable strategy for player 2 is $c$. This can be shown by iteratively removing strictly dominated strategies. For example:

1. Strategy $d$ is strictly dominated by strategy $c$ and strategy $y$ is strictly dominated by strategy $x$
2. Once we have eliminated $d$ and $y$, strategies $a$ and $b$ are strictly dominated by strategy $c$ and strategy $w$ is strictly dominated by strategy $x$
3. At this point only $x$, $z$ and $c$ remain and strategy $z$ is strictly dominated by strategy $x