Math 312, Sections 1 & 2 – Fall 2007

Solutions to the Final Exam

December 17, 2007

1. (a) (4 points) State the Intermediate Value Theorem.
See textbook.

(b) (4 points) State the Bolzano-Weierstrass Theorem.
See textbook.

2. Problem. (10 points) Consider the following function:
\[ f(x) = \frac{x^3 + 2x + 1}{x^2 + 1}, \quad x \in \mathbb{R}. \]

Using the \( \epsilon-\delta \) definition of continuity, prove that \( f \) is continuous at \( x = 0 \).

Solutions. First of all, \( f \) is defined at \( x = 0 \) and \( f(0) = 1 \). Moreover,

\[ |f(x) - f(0)| = \left| \frac{x^3 + 2x + 1}{x^2 + 1} - 1 \right| = \frac{|x^2 - x + 2|}{x^2 + 1} |x| \leq |x^2 - x + 2| \cdot |x|. \]

We must find an upper bound for \( |x^2 - x + 2| \). Suppose \( |x| < 1/2 \) then

\[ |x^2 - x + 2| \leq x^2 + |x| + 2 < \frac{1}{4} + \frac{1}{2} + 2 \leq \frac{11}{4}. \]

Let \( \epsilon > 0 \) and choose

\[ \delta = \min \left\{ \frac{1}{2}, \frac{4}{11\epsilon} \right\}. \]

It follows

\[ |f(x) - f(0)| \leq \frac{11}{4} |x| < \epsilon \quad \text{whenever} \quad |x| < \delta \]

3. Problem. (12 points) Let

\[ S_1 = \left\{ \frac{1}{n} + \cos \frac{n\pi}{2} : n \in \mathbb{N} \right\}, \quad S_2 = \left\{ \frac{\sin(n\pi/2)}{n \cdot (n+1)} : n \in \mathbb{N} \right\}. \]
For each set, determine the sup, inf, min, max, if they exist. (Prove that your answer is correct.)

**Solutions.** a) We will show that

\[ \sup S_1 = \max S_1 = \frac{5}{4}, \quad \inf S_1 = -1. \]

We have:

\[ \frac{1}{n} - 1 \leq \frac{1}{n} + \cos \frac{n\pi}{2} \leq \frac{1}{n} + 1. \]

We deduce that \( \inf S_1 = -1 \). After computations, one sees

\[ \frac{1}{n} + \cos \frac{n\pi}{2} < \frac{5}{4} \quad \text{for } n = 1, 2, 3. \]

For \( n = 4 \), one obtains

\[ \frac{1}{n} + \cos \frac{n\pi}{2} = \frac{5}{4} \]

The sequence \( \left\{ \frac{1}{n} \right\} \) is decreasing. Thus, one gets

\[ \frac{1}{n} + \cos \frac{n\pi}{2} \leq \frac{5}{4} \quad \text{for all } n \geq 1. \]

Thus, \( \sup S_1 = \max S_1 = \frac{5}{4} \) and \( \inf S_1 = -1 \). The minimum of \( S_1 \) does not exist.

b) We will show that

\[ \sup S_2 = \max S_2 = \frac{1}{2}, \quad \inf S_2 = \min S_2 = -\frac{1}{12}. \]

For all \( n \in \mathbb{N} \), one has

\[ \frac{-1}{n \cdot (n + 1)} \leq \frac{\sin(n\pi/2)}{n \cdot (n + 1)} \leq \frac{1}{n \cdot (n + 1)} \leq \frac{1}{2}. \]

Furthermore the sequence \( \left\{ \frac{-1}{n \cdot (n+1)} \right\} \) is increasing. One gets

\[ \frac{\sin(n\pi/2)}{n \cdot (n + 1)} > -\frac{1}{12} \quad \text{for all } n \neq 3 \quad \text{and} \quad \frac{\sin(3\pi/2)}{3(4)} = -\frac{1}{12}. \]

QED
4. Problem. (12 points) Find the radius of convergence of the following power series and determine whether it converges at the end points.

(a) \( \sum_{n=1}^{\infty} \frac{1}{5^n \sqrt{n}} x^n \)

(b) \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n!)} x^n \)

Solutions. Finding the radius of convergence for the above power series:

Let (a) \( b_n = \frac{x^n}{5^n \sqrt{n}} \). Then

\[
\frac{|b_{n+1}|}{b_n} = \frac{|x|}{5} \sqrt{\frac{n}{n+1}}
\]

It follows

\[
\lim_{n \to \infty} \frac{|b_{n+1}|}{b_n} = \frac{|x|}{5}.
\]

The power series converges if \( |x| < 5 \), that is, \(|x| < 5\). It diverges when \(|x| > 5\). Hence the radius of convergence is \( R = 5 \).

Convergence at the endpoints for the above power series.

- At \( x = -5 \), one gets the series

\[
\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.
\]

By Cauchy’s alternating series test, this series is conditionally convergent.

- At \( x = 5 \), one gets the series

\[
\sum_{n=1}^{\infty} \frac{5^n}{5^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.
\]

This is a \( p \)-series with \( p = 1/2 < 1 \). The series is divergent.

(b) Consider the power series

\[
\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n.
\]

Let \( b_n = \frac{(n!)^2}{(2n)!} x^n \). Then

\[
\frac{|b_{n+1}|}{b_n} = \frac{(n+1)!}{(n!)^2} \cdot \frac{(2n)!}{(2(n+1))!} \cdot |x| = \frac{(n+1)^2}{(2n+1)(2n+2)} |x|.
\]
One gets
\[ \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \frac{|x|}{4} \]
The power series converges if \( \frac{|x|}{4} < 1 \), that is \( |x| < 4 \). It diverges when \( |x| > 4 \). Hence the radius of convergence is \( R = 4 \).

**Convergence at the endpoints for the above power series.**

- At \( x = 4 \), one gets the series
  \[ \sum_{n=1}^{\infty} \frac{4^n(n!)^2}{(2n)!} \].
  Let \( a_n = \frac{4^n(n!)^2}{(2n)!} \) then
  \[ \frac{a_{n+1}}{a_n} = \frac{4(n + 1)^2}{(2n + 2)(2n + 1)} = \frac{2n + 2}{2n + 1} > 1 \]
  Since the ratio is greater than 1, the positive sequence \( \{a_n\} \) is strictly increasing. Consequently, \( \lim a_n \neq 0 \). By the \( n \)-th term Test, the series diverges.

- At \( x = -4 \), one gets the series
  \[ \sum_{n=1}^{\infty} \frac{(-4)^n(n!)^2}{(2n)!} \].
  We cannot apply Cauchy’s test because
  \[ \lim_{n \to \infty} \left| \frac{(-4)^n(n!)^2}{(2n)!} \right| \neq 0. \]
  But, the \( n \)-th term Test ensures that the series diverges.

QED

5. **Problem.** Let \( \{a_n\} \) be a sequence of real numbers such that
  \[ |a_{n+1} - a_n| = \frac{2}{3} |a_n - a_{n-1}|, \quad \text{for} \quad n \geq 1 \]
  a) (4 points) Suppose that \( a_1 = 7 \) and \( a_0 = 4 \). Show that
  \[ |a_{n+1} - a_n| = \frac{2^n}{3^{n-1}} \]
b) (6 points) Prove that \( \{a_n\} \) is a Cauchy sequence.
c) (2 points) Is \( \{a_n\} \) convergent? Justify your answer.

Solutions.
a) We will prove by induction that \(|a_{n+1} - a_n| = \frac{2^n}{3^{n-1}}\).

- For \( n = 1 \), one has: \(|a_2 - a_1| = \frac{2}{3}|a_1 - a_0| = \frac{2}{3}|7 - 4| = \frac{2}{3} = 2\)

- Suppose \(|a_{n+1} - a_n| = \frac{2^n}{3^{n-1}} \) for some \( n > 1 \). Then

  \(|a_{n+2} - a_{n+1}| = \frac{2}{3}|a_{n+1} - a_n| = \frac{2}{3} \cdot \frac{2^n}{3^{n-1}} = \frac{2^{n+1}}{3^n}\)

- Therefore, \(|a_{n+1} - a_n| = \frac{2^n}{3^{n-1}} \) for all \( n \in \mathbb{N} \).

b) Suppose \( m > n \) then

\[ |a_m - a_n| \leq |a_m - a_{m-1}| + \cdots + |a_{n+1} - a_n| = \left( \frac{2^{m-1}}{3^{m-1}} + \cdots + \frac{2^n}{3^n} \right) |a_1 - a_0| \]

It follows that

\[ |a_m - a_n| \leq \frac{2^n}{3^n} \left( \frac{2^{m-n-1}}{3^{m-n-1}} + \cdots + \frac{2}{3} + 1 \right) |a_1 - a_0| \leq \frac{2^n}{3^n} \left( \frac{1}{1 - 2/3} \right) |a_1 - a_0| = \frac{2^n}{3^{n-1}} |a_1 - a_0| \]

Given \( \epsilon > 0 \), one has

\[ |a_m - a_n| \leq \frac{2^n}{3^{n-1}} |a_1 - a_0| < \epsilon \quad \text{for} \quad m > n \gg 1 \]

This shows that \( \{a_n\} \) is a Cauchy sequence.

QED

c) The sequence \( \{a_n\} \) is convergent since all Cauchy sequences of real numbers are convergent.

6. Problem.
a) (3 points) Show that the following sequence \( \{a_n\} \) is bounded above by

\[ a_n = \frac{\sqrt{(2n + 1)(2n + 3)}}{2n + 2}, \quad n \in \mathbb{N} \]

b) (8 points) Let

\[ b_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \sqrt{2n + 1}, \quad n \in \mathbb{N} \]
Prove that the sequence \( \{b_n\} \) is decreasing. (Hint: show that \( \frac{b_{n+1}}{b_n} \leq 1 \)).

c) (3 points) Is \( \{b_n\} \) convergent? Justify your answer.

**Solutions.** For all \( n \in \mathbb{N} \), one has:

\[
a_n^2 = \frac{(2n + 1)(2n + 3)}{(2n + 2)^2}
\]

Since \( 4n^2 + 8n + 3 < 4n^2 + 8n + 4 \), it follows \( a_n^2 < 1 \). Since \( a_n > 0 \), we deduce that \( a_n < 1 \) for all \( n \in \mathbb{N} \).

b) We have:

\[
\frac{b_{n+1}}{b_n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots 2n + 2} \cdot \frac{\sqrt{2n + 3}}{\sqrt{2n + 1}}
\]

Hence the sequence \( \{b_n\} \) is strictly decreasing.

c) The sequence \( \{b_n\} \) converges by the Completeness Theorem since it is decreasing and positive.

7. **Problem.** (9 points) Suppose that \( 0 < y \leq \frac{1}{e^2} \). Prove that, for each value of \( y \), the equation

\[
\ln x - x^2 y = 0
\]

has a solution \( x \) lying in the interval \((1, e]\).

**Solutions.** Notice that \( x = e \) is a solution of the equation \( \ln x - x^2 y = 0 \) when \( y = \frac{1}{e^2} \). Now suppose that \( 0 < y < \frac{1}{e^2} \). For each fixed value of \( y \), we set

\[
f_y(x) = \ln x - x^2 y
\]

The function \( f_y(x) \) is continuous on \( \mathbb{R} \) since it is the difference of two continuous functions. More over \( f(1) = -y < 0 \) and \( f(e) = 1 - e^2 y > 0 \). By the Intermediate Value Theorem, there is a a solution \( x \) lying in \((1, e]\).

8. **Problem.**

a) (6 points) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that

\[
|f(x) - f(y)| \leq (x - y)^4 \quad \text{for all } x, y \in \mathbb{R}.
\]
Prove that its derivative is null everywhere. Hint: use the Squeeze Theorem.

b) (3 points) Using the Mean Value Theorem, show that $f$ is a constant function.

**Solutions.** Suppose $x \neq y$, then

$$|f(x) - f(y)| \leq (x - y)^4 \quad \text{implies} \quad 0 \leq \left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|^3$$

By the Squeeze Theorem, one gets

$$0 \leq |f'(y)| \leq \lim_{x \to y} |x - y|^3 = 0.$$ 

Thus, $f'(y) = 0$ for all $y \in \mathbb{R}$. Now, we will show that $f$ is constant. Let $y$ be an arbitrary nonzero real number. By the Mean Value Theorem, there exists a number $c$ between $y$ and the origin such that

$$f(y) - f(0) = y \cdot f'(c) = 0,$$

since $f'(c) = 0$. Hence, $f(y) = f(0)$ for all point $y$ in the real line.

9. **Problem.**

a) (4 points) Is the following function differentiable at the origin? Justify your answer.

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$$

(b) (10 points) Let $f(x) = \sin \frac{1}{x^2}$. Show that $f$ is NOT uniformly continuous on $(0, 1]$.

**Solutions.** (a) One has:

$$\lim_{x \to 0^+} \frac{f(x) - f(1)}{x} = \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty$$

Therefore, the function is differentiable at $x = 0$.

(b) Consider the sequence

$$x_n = \sqrt{\frac{2}{\pi + 4\pi n}} \quad y_n = \sqrt{\frac{2}{3\pi + 4\pi n}}$$
One gets

\[ |x_n - y_n| = \left| \sqrt{\frac{2}{\pi + 4\pi n}} - \sqrt{\frac{2}{3\pi + 4\pi n}} \right| \]

\[ = \frac{2\sqrt{2} \pi}{(\sqrt{\pi + 4\pi n} + \sqrt{3\pi + 4\pi n}) \sqrt{(\pi + 4\pi n)(3\pi + 4\pi n)}} \]

It follows that \( |x_n - y_n| \to 0 \) as \( n \to \infty \). On the other hand, \( f(x_n) = 1 \) and \( f(y_n) = -1 \). Thus,

\[ |f(x_n) - f(y_n)| = 2. \]

Pick \( \epsilon = 1 \). For any arbitrary positive number \( \delta \), there is an integer \( n \) such that \( |x_n - y_n| < \delta \) but \( |f(x_n) - f(y_n)| = 2 > \epsilon \) This shows that \( f \) is NOT uniformly continuous on \((0, 1]\).