14. Differentiation

Definition. Let $D$ be a subset of $\mathbb{R}$ containing a neighborhood of $a$, that is, an open interval $(a-r, a+r)$, with $r > 0$. We say that a function $f : D \to \mathbb{R}$ is differentiable at $a$ if the following limit exists:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

The real number $f'(a)$ is called the derivative of $f$ at $a$. The quotient

$$\frac{f(x) - f(a)}{x - a}$$

is referred to as the difference quotient. Equivalently, setting $x = a + \Delta x$, one gets

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = f'(a).$$

Example 1. Let $f(x) = x^2$ on $\mathbb{R}$. Show that $f$ is differentiable at each point $a \in \mathbb{R}$.

Solution We have:

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a} = x + a$$

Therefore,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 2a.$$ 

In other words $f'(a) = 2a$, for all $a$ in $\mathbb{R}$.

Example 2. Let $f(x) = \sqrt{x}$ on the open interval $I = (0, \infty)$. Prove that $f$ is differentiable at every point of $I$.

Solution For every $a \in I$, we have:

$$\frac{f(x) - f(a)}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x-a)(\sqrt{x} + \sqrt{a})} = \frac{x-a}{(x-a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}$$
It follows
\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \frac{1}{2\sqrt{a}} \]
Thus, \( f'(a) = \frac{1}{2\sqrt{a}} \), for all \( a \) in \( \mathbb{R} \).

**Remark.** Assume that the following right-hand and left-hand limits exist.
\[ \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = f'(a^+), \quad \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a^-). \]
The function \( f \) is differentiable at \( a \) if and only if \( f'(a^+) \) and \( f'(a^-) \) exist and are the same. Then, \( f'(a) = f'(a^-) = f'(a^+) \).

**Example 3.** Let \( f(x) = |x| \) defined on \( \mathbb{R} \). Is \( f \) differentiable at zero?

**Solution** We have:
\[ \lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1. \]
Since the right-hand and the left-hand limits of \( f \) at zero are different, \( f \) is not differentiable at zero.

**Example 4.** Let \( f(x) = x \sin \frac{1}{x} \) for \( x \neq 0 \) and \( f(0) = 0 \). Prove that \( f \) is not differentiable at zero.

**Solution.** We have:
\[ \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \to 0} \frac{\sin \frac{1}{x}}{x}. \]
The limit does not exist. Hence \( f \) is not differentiable at zero.

**Local extremum points**

**Definition.** Let \( f : I \to \mathbb{R} \) be a function on \( I \), where \( I \) is an open interval.
- A point \( c \) in \( I \) is a local maximum of \( f \) if \( f(c) \geq f(x) \) for \( x \approx c \).
- A point \( c \) in \( I \) is a local minimum of \( f \) if \( f(c) \leq f(x) \) for \( x \approx c \).

**Theorem.** Let \( f \) be a differentiable function on an open interval \( I \). If \( a \) is a local extremum point then \( f'(a) = 0 \).

**Remark.** The converse of the previous theorem is false. For example, if \( f(x) = x^3 \) then \( f'(0) = 0 \) but \( a = 0 \) is not a local extremum point.