### Static vs. Dynamic Models

- The **classical regression model** of Chapter 2 is a static model since a value $x_t$ only depends on $t$ and not $x_{t-1}, x_{t-2}, ...$
- The **random walk model** is dynamic since computation of $x_t$ uses $x_{t-1}$, however this model was not stationary (recall its covariance function is given by $\gamma(s, t) = \min(s, t) \sigma^2$).
- This leads us to a large class of dynamic models which are stationary — the **ARMA models**.
- Properties of the random walk model and the ARMA model are combined in the (nonstationary) **ARIMA model** (§3.7, 3.8, 3.9).

### AR Example

Consider the following AR(1) process

$$x_t = -0.9x_{t-1} + w_t$$

where $w_t \sim \mathcal{N}(0, 1)$. We simulate the process in R.

```r
> plot(arima.sim(list(order=c(1,0,0), ar=-0.9), n=100))
```
**Definition (AR(p) Model)**

An autoregressive model of order $p$ is of the form

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$$

which has mean $\mu$ given by

$$\mu = \frac{\alpha}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

(assuming the denominator is nonzero).

Notice that $x_t$ is auto-regressed on $x_{t-1}, \ldots, x_{t-p}$ however these regressors have random components, whereas the previous theory had fixed components $z_t$.

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**AR(1) — MA(∞) Representation**

Let's investigate a general AR(1) process given by

$$x_t = \phi x_{t-1} + w_t$$

Iterating backwards gives

$$x_t = \phi x_{t-1} + w_t$$

$$= \phi (\phi x_{t-2} + w_{t-1}) + w_t$$

$$= \phi^2 x_{t-2} + \phi w_{t-1} + w_t$$

$$= \phi^2 (\phi x_{t-3} + w_{t-2}) + \phi^2 w_{t-1} + w_t$$

$$= \phi^3 x_{t-3} + \phi^2 w_{t-2} + \phi w_{t-1} + w_t$$

$$\vdots$$

$$= \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j}$$

Provided $|\phi| < 1$, we can represent the AR(1) model as

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$$

In this representation, we easily see

$$\mathbb{E}(x_t) = \sum_{j=0}^{\infty} \phi^j \mathbb{E}(w_{t-j}) = 0,$$

and the autocovariance function is computed as ...
AR(1) — ACF Computation

\[\gamma(h) = \text{cov}(x_{t+h}, x_t)\]
\[= E \left( \left( \sum_{j=0}^{\infty} \phi^j w_{t+j} \right) \left( \sum_{k=0}^{\infty} \phi^k w_{t-k} \right) \right)\]
\[= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \phi^j \phi^k E(w_{t+j}w_{t-k})\]
[nonzero only when \(h-j=-k\)]
\[= \sigma^2 \phi |h| \sum_{k=0}^{\infty} \phi^{2k}\]
\[= \frac{\sigma^2 \phi |h|}{1 - \phi^2}\]

\[\rho(h) = \frac{\gamma(h)}{\gamma(0)}\]
\[= \frac{\sigma^2 \phi |h|}{1 - \phi^2} \cdot \frac{1 - \phi^2}{\sigma^2}\]
\[= \phi |h|\]

Note that \(\rho(h)\) satisfies the recursion
\[\rho(h) = \phi \rho(h-1)\]
(with the usual initial constraint \(\rho(0) = 1\)).

Causality

From the AR(1) model
\[x_t = \phi x_{t-1} + w_t\]
we can solve for \(x_{t-1}\) as
\[x_{t-1} = \frac{1}{\phi} x_t + \frac{1}{\phi} w_t = \frac{1}{\phi} x_t + \bar{w}_t\]
If \(|\phi| > 1\), then the above expression gives a stationary representation of \(x_t\) in terms of future values of \(x_t\). This is useless since we don’t know the future!

Definition (Causality)

When a process does not depend on the future, such as the AR(1) with \(|\phi| < 1\), the process is called causal.

Moving Average Model — MA(q)

Definition (Moving average model — MA(q))
The moving average model of order \(q\) is defined to be
\[x_t = \mu + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}\]
where \(\theta_1, \theta_2, \ldots \theta_q\) are parameters in \(\mathbb{R}\).

The above model can be compactly written as
\[x_t = \mu + \theta(B) w_t\]
where \(\theta(B)\) is the moving average operator.

Definition (Moving Average Operator)
The moving average operator is
\[\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q\]
Consider the mean zero MA(1) process
\[ x_t = w_t + \theta w_{t-1} \]

Then the autocovariance function is computed as
\[
\gamma(h) = \text{cov}(x_{t+h}, x_t) = \mathbb{E} \left[ (w_{t+h} + \theta w_{t+h-1})(w_t + \theta w_{t-1}) \right]
\]
\[
= \mathbb{E}(w_{t+h}w_t) + \theta \mathbb{E}(w_{t+h-1}w_t) + \theta \mathbb{E}(w_{t+h}w_{t-1}) + \theta^2 \mathbb{E}(w_{t+h-1}w_{t-1})
\]
\[
= \begin{cases} 
(1 + \theta^2)\sigma^2, & h = 0 \\
\theta \sigma^2, & |h| = 1 \\
0, & |h| > 1
\end{cases}
\]

Therefore
\[
\rho(h) = \begin{cases} 
1, & h = 0 \\
\frac{\theta}{1 + \theta^2}, & |h| = 1 \\
0, & |h| > 1
\end{cases}
\]

Note that when \( \theta = \frac{1}{\sigma^2} \), we have
\[
\rho(h) = \begin{cases} 
1, & h = 0 \\
\frac{\theta^{h-1}}{1 + \theta^{h-2}}, & |h| = 1 \\
0, & |h| > 1
\end{cases}
\]

MA(q) process:
\[ x_t = \mu + \phi_1 x_{t-1} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q} \]

Definition (Invertibility)
An MA model process is called invertible if \( x_t \) has the AR(\( \infty \)) representation
\[ x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} + w_t \]

In the MA(1) case, this amounts to \( |\theta| < 1 \).
Facts:
- The ARMA($p$, $q$) model given by $\phi(B)x_t = \theta(B)w_t$ is causal if and only if $|\phi(z)| \neq 0$ when $|z| \leq 1$
  i.e. all roots including complex roots of $\phi(z)$ lie outside the unit circle.
- The ARMA($p$, $q$) model given by $\phi(B)x_t = \theta(B)w_t$ is invertible if and only if $|\theta(z)| \neq 0$ when $|\theta| \leq 1$
  i.e. all roots including complex roots of $\theta(z)$ lie outside the unit circle.

Example – Removing Redundancy

Consider the process
\[
x_t = .4x_{t-1} + .45x_{t-2} + w_t + w_{t-1} + .25w_{t-2}
\]
which is equivalent to
\[
(1 - .4B - .45B^2) x_t = (1 + B + .25B^2) w_t
\]
\[
\phi(B) \quad \theta(B)
\]

Is this process really ARMA(2,2)? No! Factoring $\phi(z)$ and $\theta(z)$ gives
\[
\phi(z) = 1 - .4z - .45z^2 = (1 + .5z)(1 - .9z)
\]
\[
\theta(z) = 1 + z + .25z^2 = (1 + .5z)^2
\]

Removing the common term $(1 + .5z)$ gives the reduced model
\[
x_t = .9x_{t-1} + .5w_{t-1} + w_t
\]

Example – Causality and Invertibility

The reduced model is $\phi(B)x_t = \theta(B)w_t$ where
\[
\phi(z) = 1 - .9z
\]
\[
\theta(z) = 1 + .5z
\]

- There is only one root of $\phi(z)$ which is $z = 10/9$, and $|10/9|$ lies outside the unit circle so this ARMA model is causal.
- There is only one root of $\theta(z)$ which is $z = -2$, and $|-2|$ lies outside the unit circle so this ARMA model is invertible.
Example – MA(∞) Representation

Example (cont.)

Note that

$$\psi(z) = \frac{\theta(z)}{\phi(z)} = \frac{1 + .5z}{1 - .9z}$$

$$= (1 + .5z)(1 + .9z + .9^2z^2 + .9^3z^3 + \cdots) \quad \text{for } |z| \leq 1$$

$$= 1 + (.5 + .9)z + (.5(.9) + .9^2)z^2 + (.5(.9^2) + .9^3)z^3 + \cdots \quad \text{for } |z| \leq 1$$

$$= 1 + (.5 + .9)\sum_{j=1}^{\infty} .9^{j-1}z^j \quad \text{for } |z| \leq 1$$

This gives the following MA(∞) representation:

$$x_t = \frac{\theta(B)}{\phi(B)}w_t = \left(1 + 1.4\sum_{j=1}^{\infty} .9^{j-1}B^j\right)w_t = w_t + 1.4\sum_{j=1}^{\infty} .9^{j-1}w_{t-j}$$

Example – AR(∞) Representation

Example (cont.)

Similarly, one can show

$$\pi(z) = \frac{\phi(z)}{\theta(z)} = \frac{1 - .9z}{1 + .5z}$$

$$= 1 - .9z + .5z^2 + .9(1.4)z^3 + \cdots$$

$$= 1 + 1.4\sum_{j=1}^{\infty} (-.5)^{j-1}z^j \quad \text{for } |z| \leq 1$$

This gives the following AR(∞) representation:

$$x_t = 1.4\sum_{j=1}^{\infty} (-.5)^{j-1}w_{t-j} + w_t$$

Textbook Reading

Read the following sections from the textbook

- §3.3 (Difference Equations)
- §3.4 (Autocorrelation and Partial Autocorrelation Functions)

Textbook Problems

Do the following exercise from the textbook

- 3.1
- 3.3
- 3.4