Yule-Walker Equations and Moving Average Models

George Udny Yule
(1871-1951)

Sir Gilbert Thomas Walker
(1868-1958)

"Every cell phone call solves the Yule-Walker equations every ten microseconds."
Thierry Dutoit

"On a Method of Investigating Periodicities in Disturbed Series, with Special Reference to Wolfer's Sunspot Numbers" (1927)

"On Periodicity in Series of Related Terms" (1931)
Outline

1 §7.1: Yule-Walker Equations

2 §3.2 Moving Average Processes

3 Homework 2c
Yule-Walker Equations

Start with the mean zero AR\((p)\) model:

\[
Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \tag{\star}
\]

Multiply both sides of \((\star)\) by \(Z_{t-h}\) for \(h = 1, \ldots, p\):

\[
Z_t Z_{t-h} = \phi_1 Z_{t-1} Z_{t-h} + \phi_2 Z_{t-2} Z_{t-h} + \cdots + \phi_p Z_{t-p} Z_{t-h} + a_t Z_{t-h} \tag{\star\star}
\]

Take expectations throughout:

\[
\gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \tag{1}
\]

Now take the expectation of \((\star\star)\) with \(h = 0\):

\[
\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \mathbb{E}(Z_t a_t) \tag{\star\star\star}
\]

Rearranging \((\star\star\star)\) gives

\[
\sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \cdots - \phi_p \gamma(p) \tag{2}
\]

Equations (1) and (2) are the Yule-Walker Equations.
Yule-Walker Equations

Start with the mean zero AR($p$) model:

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \tag{\star}$$

Multiply both sides of (\star) by $Z_{t-h}$ for $h = 1, \ldots, p$:

$$Z_t Z_{t-h} = \phi_1 Z_{t-1} Z_{t-h} + \phi_2 Z_{t-2} Z_{t-h} + \cdots + \phi_p Z_{t-p} Z_{t-h} + a_t Z_{t-h} \tag{\star\star}$$

Take expectations throughout:

$$\gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \tag{1}$$

Now take the expectation of (\star\star) with $h = 0$:

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \mathbb{E}(Z_t a_t) \quad \tag{\star\star\star}$$

Rearranging (\star\star\star) gives

$$\sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \cdots - \phi_p \gamma(p) \tag{2}$$

Equations (1) and (2) are the Yule-Walker Equations.
Yule-Walker Equations

Start with the mean zero AR($p$) model:

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \quad (\star)$$

Multiply both sides of (\star) by $Z_{t-h}$ for $h = 1, \ldots, p$:

$$Z_t Z_{t-h} = \phi_1 Z_{t-1} Z_{t-h} + \phi_2 Z_{t-2} Z_{t-h} + \cdots + \phi_p Z_{t-p} Z_{t-h} + a_t Z_{t-h} \quad (\star\star)$$

Take expectations throughout:

$$\gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \quad (1)$$

Now take the expectation of (\star\star) with $h = 0$:

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \mathbb{E}(Z_t a_t) \quad (\star\star\star)$$

Rearranging (\star\star\star) gives

$$\sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \cdots - \phi_p \gamma(p) \quad (2)$$

Equations (1) and (2) are the Yule-Walker Equations.
Yule-Walker Equations

Start with the mean zero AR($p$) model:

\[ Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \]  

$(\star)$

Multiply both sides of $(\star)$ by $Z_{t-h}$ for $h = 1, \ldots, p$:

\[ Z_t Z_{t-h} = \phi_1 Z_{t-1} Z_{t-h} + \phi_2 Z_{t-2} Z_{t-h} + \cdots + \phi_p Z_{t-p} Z_{t-h} + a_t Z_{t-h} \]  

$(\star\star)$

Take expectations throughout:

\[ \gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \cdots + \phi_p \gamma(h-p) \]  

$(1)$

Now take the expectation of $(\star\star)$ with $h = 0$:

\[ \gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \mathbb{E}(Z_t a_t) \]  

$(\star\star\star)$

Rearranging $(\star\star\star)$ gives

\[ \sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \cdots - \phi_p \gamma(p) \]  

$(2)$

Equations $(1)$ and $(2)$ are the Yule-Walker Equations.
Yule-Walker Equations

Start with the mean zero AR($p$) model:

\[ Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \]  

(*)

Multiply both sides of (*) by $Z_{t-h}$ for $h = 1, \ldots, p$:

\[ Z_t Z_{t-h} = \phi_1 Z_{t-1} Z_{t-h} + \phi_2 Z_{t-2} Z_{t-h} + \cdots + \phi_p Z_{t-p} Z_{t-h} + a_t Z_{t-h} \]  

(**)

Take expectations throughout:

\[ \gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \]  

(1)

Now take the expectation of (**) with $h = 0$:

\[ \gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \mathbb{E}(Z_t a_t) \]  

(***)

Rearranging (***) gives

\[ \sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \cdots - \phi_p \gamma(p) \]  

(2)

Equations (1) and (2) are the Yule-Walker Equations.
Yule-Walker Equations

Start with the mean zero AR(p) model:

\[ Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t \]  

\[(*)\]

Multiply both sides of (*) by \( Z_{t-h} \) for \( h = 1, \ldots, p \):

\[ Z_t Z_{t-h} = \phi_1 Z_{t-1} Z_{t-h} + \phi_2 Z_{t-2} Z_{t-h} + \cdots + \phi_p Z_{t-p} Z_{t-h} + a_t Z_{t-h} \]  

\[(**)\]

Take expectations throughout:

\[ \gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \]  

\[(1)\]

Now take the expectation of \((**)\) with \( h = 0 \):

\[ \gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \cdots + \phi_p \gamma(p) + \mathbb{E}(Z_t a_t) \]  

\[(***)\]

Rearranging \((***)\) gives

\[ \sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \cdots - \phi_p \gamma(p) \]  

\[(2)\]

Equations (1) and (2) are the Yule-Walker Equations.
Y-W Equations in Matrix Form

From the recurrence

\[ \gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \quad (1), \]

extract the \( p \) equations

\[
\begin{align*}
\gamma(1) &= \phi_1 \gamma(0) + \phi_2 \gamma(1) + \cdots + \phi_p \gamma(p - 1) \\
\gamma(2) &= \phi_1 \gamma(1) + \phi_2 \gamma(0) + \cdots + \phi_p \gamma(p - 2) \\
&\vdots \\
\gamma(p) &= \phi_1 \gamma(p - 1) + \phi_2 \gamma(p - 2) + \cdots + \phi_p \gamma(0)
\end{align*}
\]

\[
\begin{pmatrix}
\gamma(1) \\
\gamma(2) \\
\vdots \\
\gamma(p)
\end{pmatrix}
= 
\begin{pmatrix}
\gamma(0) & \gamma(1) & \cdots & \gamma(p - 1) \\
\gamma(1) & \gamma(0) & \cdots & \gamma(p - 2) \\
\vdots & \vdots & \ddots & \vdots \\
\gamma(p - 1) & \gamma(p - 2) & \cdots & \gamma(0)
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_p
\end{pmatrix}
\]

Hence \( \Gamma_p \phi = \gamma_p \) where \( \Gamma_p = \{\gamma(k - j)\}_{i,j=1}^p \).
Y-W Equations in Matrix Form

From the recurrence

$$\gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \quad (1),$$

extract the $p$ equations

$$\begin{align*}
\gamma(1) &= \phi_1 \gamma(0) + \phi_2 \gamma(1) + \cdots + \phi_p \gamma(p - 1) \\
\gamma(2) &= \phi_1 \gamma(1) + \phi_2 \gamma(0) + \cdots + \phi_p \gamma(p - 2) \\
\vdots \hspace{5cm} & \hspace{5cm} \\
\gamma(p) &= \phi_1 \gamma(p - 1) + \phi_2 \gamma(p - 2) + \cdots + \phi_p \gamma(0)
\end{align*}$$

$$\begin{pmatrix}
\gamma(1) \\
\gamma(2) \\
\vdots \\
\gamma(p)
\end{pmatrix}
= 
\begin{pmatrix}
\gamma(0) & \gamma(1) & \cdots & \gamma(p - 1) \\
\gamma(1) & \gamma(0) & \cdots & \gamma(p - 2) \\
\vdots & \vdots & \ddots & \vdots \\
\gamma(p - 1) & \gamma(p - 2) & \cdots & \gamma(0)
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_p
\end{pmatrix}$$

Hence $\Gamma_p \phi = \gamma_p$ where $\Gamma_p = \{ \gamma(k - j) \}_{i,j=1}^p$. 
Y-W Equations in Matrix Form

From the recurrence

\[ \gamma(h) = \phi_1 \gamma(h - 1) + \phi_2 \gamma(h - 2) + \cdots + \phi_p \gamma(h - p) \]  

\[ (1), \]

extract the \( p \) equations

\[
\begin{align*}
\gamma(1) &= \phi_1 \gamma(0) + \phi_2 \gamma(1) + \cdots + \phi_p \gamma(p - 1) \\
\gamma(2) &= \phi_1 \gamma(1) + \phi_2 \gamma(0) + \cdots + \phi_p \gamma(p - 2) \\
\vdots &\quad \\
\gamma(p) &= \phi_1 \gamma(p - 1) + \phi_2 \gamma(p - 2) + \cdots + \phi_p \gamma(0)
\end{align*}
\]

\[
\begin{pmatrix}
\gamma(1) \\
\gamma(2) \\
\vdots \\
\gamma(p)
\end{pmatrix}
= 
\begin{pmatrix}
\gamma(0) & \gamma(1) & \cdots & \gamma(p - 1) \\
\gamma(1) & \gamma(0) & \cdots & \gamma(p - 2) \\
\vdots & \vdots & \ddots & \vdots \\
\gamma(p - 1) & \gamma(p - 2) & \cdots & \gamma(0)
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_p
\end{pmatrix}
\]

Hence \( \Gamma_p \phi = \gamma_p \) where \( \Gamma_p = \{\gamma(k - j)\}_{i,j}^{p} \).
Under the method of moments approach, we estimate $\phi = (\phi_1, \phi_2, \ldots, \phi_p)$ with

$$\hat{\phi} = \hat{\Gamma}_p^{-1} \hat{\gamma}_p$$

and also

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}_1 \hat{\gamma}(1) - \hat{\phi}_2 \hat{\gamma}(2) - \cdots - \hat{\phi}_p \hat{\gamma}(p)$$

$$= \hat{\gamma}(0) - \hat{\phi}' \hat{\gamma}_p$$

$$= \hat{\gamma}(0) - \left( \hat{\Gamma}_p^{-1} \hat{\gamma}_p \right)' \hat{\gamma}_p$$

$$= \hat{\gamma}(0) - \hat{\gamma}_p' \hat{\Gamma}_p^{-1} \hat{\gamma}_p$$

where $\hat{\Gamma}_p = \{\hat{\gamma}(k - j)\}_{j,k=1}^p$ and $\hat{\gamma}_p = (\hat{\gamma}(1), \ldots, \hat{\gamma}(p))'$. 
Estimation Using the Y-W Equations

Under the method of moments approach, we estimate $\phi = (\phi_1, \phi_2, \ldots, \phi_p)$ with

$$\hat{\phi} = \hat{\Gamma}_p^{-1} \hat{\gamma}_p$$

and also

$$\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}_1 \hat{\gamma}(1) - \hat{\phi}_2 \hat{\gamma}(2) - \cdots - \hat{\phi}_p \hat{\gamma}(p)$$

$$= \hat{\gamma}(0) - \hat{\phi}' \hat{\gamma}_p$$

$$= \hat{\gamma}(0) - \left(\hat{\Gamma}_p^{-1} \hat{\gamma}_p\right)' \hat{\gamma}_p$$

$$= \hat{\gamma}(0) - \hat{\gamma}_p' \hat{\Gamma}_p^{-1} \hat{\gamma}_p$$

where $\hat{\Gamma}_p = \{\hat{\gamma}(k - j)\}_{j,k=1}^p$ and $\hat{\gamma}_p = (\hat{\gamma}(1), \ldots, \hat{\gamma}(p))'$. 
Outline

1. §7.1: Yule-Walker Equations
2. §3.2 Moving Average Processes
3. Homework 2c
Moving Average Model — MA(q)

Definition (Moving average model — MA(q))

The moving average model of order \( q \) is defined to be

\[
Z_t = \mu + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}
\]

where \( \theta_1, \theta_2, \ldots \theta_q \) are parameters in \( \mathbb{R} \).

The above model can be compactly written as

\[
Z_t = \mu + \theta(B)a_t
\]

where \( \theta(B) \) is the moving average operator.

Definition (Moving Average Operator)

The moving average operator is

\[
\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q
\]
Definition (Moving average model — MA(q))

The moving average model of order \( q \) is defined to be

\[
Z_t = \mu + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}
\]

where \( \theta_1, \theta_2, \ldots, \theta_q \) are parameters in \( \mathbb{R} \).

The above model can be compactly written as

\[
Z_t = \mu + \theta(B)a_t
\]

where \( \theta(B) \) is the moving average operator.

Definition (Moving Average Operator)

The moving average operator is

\[
\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q
\]
Moving Average Model — MA($q$)

Definition (Moving average model — MA($q$))

The moving average model of order $q$ is defined to be

$$Z_t = \mu + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \cdots + \theta_q a_{t-q}$$

where $\theta_1, \theta_2, \ldots, \theta_q$ are parameters in $\mathbb{R}$.

The above model can be compactly written as

$$Z_t = \mu + \theta(B) a_t$$

where $\theta(B)$ is the moving average operator.

Definition (Moving Average Operator)

The moving average operator is

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$$
Outline

1 §7.1: Yule-Walker Equations

2 §3.2 Moving Average Processes

3 Homework 2c
Read §3.3 and §3.4.

Do exercise #3.1(a,b) and #3.5.