Bayes’ Rule

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Definition (Partition)

Events $B_1, B_2, \ldots, B_k$ are said to partition a sample space $S$ if

1. they are mutually exclusive ($B_i B_j = \emptyset$ for any pair $i$ and $j$)
2. $\bigcup_{j=1}^{k} B_j = S$
Theorem (Theorem of Total Probability)

If $B_1, B_2, \ldots, B_k$ is a partition of the sample space, then for any event $A$,

$$P(A) = \sum_{i=1}^{k} P(AB_i) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$$
Example (Exercise 3.43 slightly modified)

John flies frequently and likes to upgrade his seat to first class. He has determined that, if he checks in for his flight at least two hours early, the probability that he will get the upgrade is .75; otherwise, the probability that he will get the upgrade is .35. With his busy schedule, he checks in at least two hours before his flight only 40% of the time. What is the probability that, for a randomly selected trip, John will be able to upgrade to first class?
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\[
P(\text{upgrade}) = P(\text{arrives early})P(\text{upgrade} | \text{arrives early}) + P(\text{arrives late})P(\text{upgrade} | \text{arrives late}) = .4 \times .75 + .6 \times .35 = .51
\]
Bayes’ Rule

Theorem (Bayes’ Rule)

If the events $B_1, B_2, \ldots B_k$ form a partition of the sample space $S$, and $A$ is any event in $S$, then

\[
P(B_i|A) = \frac{P(B_iA)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}
\]

Example (Back to Exercise 3.43)

Suppose John didn’t receive an upgrade on his most recent attempt. What is the probability that he arrived late?

\[
P(\text{arrives late}|\text{no upgrade}) = \frac{P(\text{no upgrade}|\text{arrives late})P(\text{arrives late})}{1 - 0.35} \approx 80\%
\]
Theorem (Bayes’ Rule)

If the events $B_1, B_2, \ldots B_k$ form a partition of the sample space $S$, and $A$ is any event in $S$, then

$$P(B_i | A) = \frac{P(B_i A)}{P(A)} = \frac{P(A | B_i) P(B_i)}{\sum_{j=1}^{k} P(A | B_j) P(B_j)}$$

Example (Back to Exercise 3.43)

Suppose John didn’t receive an upgrade on his most recent attempt. What is the probability that he arrived late?

$$P(\text{arrives late | no upgrade}) = \frac{P(\text{no upgrade | arrives late}) P(\text{arrives late})}{P(\text{no upgrade})} = \frac{(1 - .35) \times .6}{1 - .51} \approx 80\%$$
Another Example of Bayes’ Rule

Example (Exercise 3.44)

A diagnostic test of a certain disease has 95% sensitivity and 95% specificity. Only 1% of the population has the disease in question. If the diagnostic test reports that a person chosen at random from the population tests positive, what is the conditional probability that the person does, in fact, have the disease?
Example (Exercise 3.44)

A diagnostic test of a certain disease has 95% sensitivity and 95% specificity. Only 1% of the population has the disease in question. If the diagnostic test reports that a person chosen at random from the population tests positive, what is the conditional probability that the person does, in fact, have the disease?

\[
\text{predictive value} = P(\text{disease} \mid + \text{test}) = \frac{P(+ \text{test} \mid \text{disease}) P(\text{disease})}{P(+ \text{test} \mid \text{disease}) P(\text{disease}) + P(+ \text{test} \mid \text{no disease}) P(\text{no disease})} \\
= \frac{.95 \times .01}{.95 \times .01 + (1 - .95) \times .99} \approx 16\% 
\]
Odds in favor

Definition (Odds in favor)

The **odds in favor** of an event \( A \) is

\[
\text{Odds in favor of } A = \frac{P(A)}{P(\bar{A})}
\]

Letting \( p = P(A) \), we see

\[
\text{odds} = \frac{p}{1 - p} \\
\Rightarrow (1 - p)\text{odds} = p \\
\Rightarrow p + p(\text{odds}) = 1 \\
\Rightarrow p = \frac{1}{1 + \text{odds}}
\]
Odds, Odds Ratio, and Relative Risk

<table>
<thead>
<tr>
<th></th>
<th>MI</th>
<th>No MI</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aspirin</td>
<td>139</td>
<td>10,898</td>
<td>11,037</td>
</tr>
<tr>
<td>Placebo</td>
<td>239</td>
<td>10,795</td>
<td>11,034</td>
</tr>
<tr>
<td>Total</td>
<td>378</td>
<td>21,683</td>
<td>22,071</td>
</tr>
</tbody>
</table>

Odds of suffering MI in the aspirin group is

\[
P(\text{MI}) = \frac{139}{11,037} = \frac{139}{10,898} \approx .013
\]

Odds of suffering MI in the control group is

\[
P(\text{MI}) = \frac{239}{11,034} \approx .022.
\]

The odds ratio of MI is computed as:

\[
\text{Odds ratio of MI} = \frac{\text{Odds of MI with aspirin}}{\text{Odds of MI without aspirin}} = \frac{.013}{.022} = .59
\]

The relative risk of MI is computed as:

\[
\text{relative risk of MI} = \frac{P(\text{MI}|\text{aspirin})}{P(\text{MI}|\text{placebo})} = \frac{139/11,037}{239/11,034} = .58
\]