RECURSIVE SEQUENCES IN FIRST-YEAR CALCULUS

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Abstract. This article provides ready-to-use supplementary material on recursive sequences for a second semester calculus class. It equips first-year calculus students with a basic methodical procedure based on which they can conduct a rigorous convergence or divergence analysis of many simple recursive sequences on their own without the need to invoke inductive arguments as is typically required in calculus textbooks. The sequences that are accessible to this kind of analysis are predominantly (eventually) monotonic, but also certain recursive sequences that alternate around their limit point as they converge can be considered.

1. Introduction

Numerical analysis of autonomous ordinary differential equations via time discretization and Runge-Kutta methods, as well as numerical root finding based on Newton’s method, lead to recursive sequences. These methods are relevant to many fields outside of mathematics, which matters for teaching first-year calculus because most students in these classes aim towards a degree in the sciences or in engineering, not necessarily in mathematics. Most standard calculus textbooks discuss recursive sequences only very marginally as an illustration of the Monotonic Sequence Theorem. In the process to establish monotonicity and boundedness of a particular recursive sequence, an inductive argument is typically invoked that is based on algebraic manipulations of inequalities and the particular form of the recurrence relation. Following such an argument represents a challenge for many students as it is typically their very first – and in most cases it remains their sole – exposure to mathematical induction. Even fewer calculus students will be able to work problems out on their own that require inductive arguments. This is by no means surprising because induction, or proof techniques in general, are not central topics in a typical calculus class.

While any attempt of an in-depth treatment of recursive sequences would quickly lead to a full course on dynamical systems and is therefore unreasonable within the confines of calculus, the question remains what can be done in the calculus classroom to give students more exposure to this kind of sequence, and, in particular, to give them tools that are appropriate for calculus to work out convergence or divergence problems by themselves.

In these notes I share some basic material that I developed for use in my own calculus classes to address this question. These notes represent ready-to-use supplementary material for a second semester calculus class that is otherwise based on a standard text such as [4, 7, 8] (our department uses [7]). The notes came about after I unsuccessfully tried to find such material for my classes some years

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ago. What I could find, then, went in my opinion either too far into a course on
dynamical systems, or became too theoretical, or remained a case-by-case analy-
sis that was lacking a methodical approach. Having used this material for a few
years now, I found that it enables my calculus students to comprehend recursive se-
quencies better, and, most importantly, that it equips them with a basic methodical
procedure based on which they can conduct a rigorous convergence or divergence
analysis of many simple recursive sequences on their own without the need to invoke
inductive arguments. The sequences that are accessible to this kind of analysis are
predominantly (eventually) monotonic, but also certain recursive sequences that
alternate around their limit point as they converge, such as the sequence of ratios
of subsequent Fibonacci numbers, can be considered.

The structure of this article is as follows: Sections 2 and 3 are written in a style
that is directly geared towards calculus students. Section 4 is a compilation and
adaptation of homework problems of the kind used in calculus or entry-level dynam-
ical systems textbooks as well as pertinent internet resources. Appendix A contains
rigorous analytic proofs of the arguments on which the methodical approach pre-
sented in Section 3 is based. The appendix is accessible to students attending an
introductory real analysis class.

Finally, for enrichment or additional guided independent study for the most
enthusiastic calculus students, I should mention that there are many excellent in-
troductory texts on dynamical systems that are geared towards undergraduates
such as [1, 2, 3, 5, 6] that are suitable for these purposes.

2. Graphical Analysis – Cobweb Diagrams

Consider a generic recursive sequence

\[
\begin{align*}
    a_{n+1} &= f(a_n), \quad n = 1, 2, 3, \
    a_1 &= \text{Given initial value}.
\end{align*}
\]

A cobweb diagram is a visual tool to track the behavior of the sequence \( \{a_n\} \). By
definition, each term of the sequence is obtained by evaluating the function \( f \) on
the previous term. One iterative step of passing from a sequence term \( a_n \) to the
next term \( a_{n+1} \) can be visualized as shown in Figure 1:

![Cobweb Diagram](image)

**Figure 1.** Two examples for passing from \( a_n \) to \( a_{n+1} \).

1. Graph the function \( y = f(x) \) and the diagonal \( y = x \) in the same coordinate
   system.
(2) Position a marker at the point \((a_n, a_n)\) on the diagonal. The orthogonal projection of this point to the x- or y-axis marks the location of the sequence term \(a_n\) on the axes.

(3) Draw a vertical line that connects the point \((a_n, a_n)\) on the diagonal with the function graph \(y = f(x)\), and from that intersection point draw a horizontal line that connects again with the diagonal. The so obtained point on the diagonal has coordinates \((a_{n+1}, a_{n+1})\), and thus the orthogonal projection to the axes yield the position of the next sequence term \(a_{n+1}\). If \(a_n\) is a fixed point for \(f\), i.e. if \(f(a_n) = a_n\), then \((a_{n+1}, a_{n+1}) = (a_n, a_n)\) and the sequence has become stationary.

Starting with the initial term \(a_1\), this process can be iterated in the same coordinate system and allows for a qualitative visual analysis of the first few sequence terms by following along the path of concatenated vertical and horizontal lines that alternate between the diagonal and the function graph, see Figure 2. Because the entire information is contained in that concatenated path, the sequence terms are typically not separately tracked on the x- and y-axes.

\[ a_1, a_2, a_3, a_4, a_5 \]

\[ y = f(x) \]

\[ y = x \]

**Figure 2.** Cobweb diagrams with and without tracking on the axes.

As indicated in the right diagram in Figure 2, cobweb diagrams help us to visualize nicely that the limit of a continuous recurrence must be a fixed point of the function \(f\).

### 3. Limits of Recursive Sequences

**Definition 1.** A sequence \(\{a_n\}_{n=1}^{\infty}\) is called *increasing* if \(a_n < a_{n+1}\) for all \(n \in \mathbb{N}\), and it is called *decreasing* if \(a_n > a_{n+1}\) for all \(n \in \mathbb{N}\). Sequences that are either increasing or decreasing are also called *monotonic*.

The sequence \(\{a_n\}\) is called *eventually increasing* if \(a_n < a_{n+1}\) for all \(n\) large enough, and *eventually decreasing* if \(a_n > a_{n+1}\) for all \(n\) large enough. A sequence is *eventually monotonic* if it is either eventually increasing or eventually decreasing.

Consider a recursive sequence

\[
\begin{align*}
\{a_{n+1} = f(a_n), & n = 1, 2, 3, \ldots, \\
a_1 = \text{Given initial value.}
\end{align*}
\]

We make the standing assumption that \(f\) is continuous and defined on a union of *open* intervals.
We begin by examining conditions that will insure that the sequence \( \{a_n\} \) is monotonic. To this end consider the cobweb diagrams shown in Figure 3. We restrict our attention to the interval \((a, b)\):

\[
\lim_{n \to \infty} a_n = b
\]

\[
\lim_{n \to \infty} a_n = a
\]

**Figure 3.** Monotonic recursive sequences.

(1) **No fixed point of** \( f \) **is located between** \( a \) **and** \( b \). Consequently, for \( a < x < b \), the graph of \( y = f(x) \) lies either entirely above or below the diagonal \( y = x \) as shown.

(2) **The function** \( f \) **maps the interval** \((a, b)\) **into itself.**

This can be seen in the figure from the fact that the graph of \( y = f(x) \) when restricted to \( a < x < b \) stays entirely inside the interior of the highlighted square with diagonally opposite vertices \((a, a)\) and \((b, b)\).

(3) Based on (1) and (2), we can conclude that for any choice of initial value \( a_1 \in (a, b) \) we obtain a monotonic sequence \( \{a_n\} \) that stays inside \((a, b)\).

If the graph of \( y = f(x) \) lies above the diagonal \( y = x \), that sequence is increasing with \( \lim_{n \to \infty} a_n = b \), and if the graph of \( y = f(x) \) lies below the diagonal \( y = x \) it is decreasing with \( \lim_{n \to \infty} a_n = a \).

Note that Conclusion (3) remains unchanged under both Conditions (1) and (2) if \( a \) or \( b \) (or both) are not fixed points of \( f \) themselves as depicted in Figure 3, and that \( a \) or \( b \) may also take the values \( \pm \infty \). Continuity of \( f \) on \((a, b)\) is a crucial assumption in any case.

Figure 4 shows that both Conditions (1) and (2) are generally crucial for the validity of the conclusion in (3): The graph of \( y = f(x) \) in the left diagram in Figure 4 satisfies Condition (2) but not Condition (1), while on the right Condition (2) is satisfied but not Condition (2). In both cases, the depicted cobweb diagrams each show a sequence that alternates indefinitely between two values, and consequently that sequence is not monotonic and does not have a limit (if Condition (2) fails, a sequence with generic initial value in \((a, b)\) will typically also leave the interval \((a, b)\) as indicated in Figure 4, see also Figure 5).
These observations lead to the following procedure that allows us to determine whether certain recursive sequences are eventually monotonic, and to find the limit:

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**Analyzing for monotonicity and finding the limit**

**Step 1.** Solve the fixed point equation $f(x) = x$.

**Step 2.** If $a_1$ is itself a fixed point, the sequence is constant with $a_n = a_1$ for all $n$, thus $\lim_{n \to \infty} a_n = a_1$. Otherwise, use the solutions of the fixed point equation in Step 1. and the domain intervals of $f$ to identify an open interval $(a, b)$ such that $a$ and $b$ are fixed points or domain interval endpoints (which may include $\pm \infty$) such that $a_1 \in (a, b)$, and such that $f$ does not have any fixed points in $(a, b)$.

**Step 3.** Check whether $f$ maps the interval $(a, b)$ from Step 2. into itself.

**Step 4.** If Step 3. reveals that $f$ maps $(a, b)$ into itself:

The sequence $\{a_n\}$ is monotonic, stays inside the interval $(a, b)$, and either $\lim_{n \to \infty} a_n = b$ if the sequence increases, or $\lim_{n \to \infty} a_n = a$ if it decreases. To verify which case occurs, it is enough to check whether $a_2 > a_1$ (increasing case) or $a_2 < a_1$ (decreasing case).

If Step 3. reveals that $f$ does not map $(a, b)$ into itself:

If $b > a_2 > a_1$ check whether $f$ maps $(a_1, b)$ into itself, and if $a < a_2 < a_1$ check whether $f$ maps $(a, a_1)$ into itself. If it does, the above conclusion about monotonicity and the limit remains valid.

Otherwise track some of the sequence terms and consider a **new recursive sequence** that is obtained by choosing $a_1$ among these higher terms (and the same $f$), and restart the process at Step 2. If tracking reveals that the original sequence leaves the interval $(a, b)$, choose $a_1$ to be the first term outside of $(a, b)$. Note that the new sequence is obtained from the original one simply by removing terms at the beginning, thus the limiting behavior and **eventual** monotonicity for the new sequence are the same as for the original sequence.

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**Figure 4.** The sequences alternate between $a_1$ and $a_2$. 

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[Diagram of recursive sequences and their intervals]
The cobweb diagrams shown in Figure 3 represent typical examples for cases when Step 3. determines that the function \( f \) maps the interval \((a, b)\) into itself. Figure 5 shows examples where this is not the case, but the above procedure still leads to the conclusion that the sequence is eventually monotonic and convergent. However, as the depicted example on the right in Figure 4 shows, one cannot expect that this process will generally lead to a successful conclusion.

![Cobweb diagrams](image)

**Figure 5.** Eventually monotonic recursive sequences.

**Example 1.** Consider the recursive sequence

\[
\begin{align*}
    a_{n+1} &= a_n^n, \quad n = 1, 2, 3, \ldots, \\
    a_1 &= \frac{1}{2}.
\end{align*}
\]

Show that \( \{a_n\} \) converges, and find \( \lim_{n \to \infty} a_n \).

**Solution:** Consider \( f(x) = x^x, \quad x > 0 \). Then \( a_{n+1} = f(a_n) \) for all \( n = 1, 2, 3, \ldots \)

Step 1. Find the fixed points:

\[
x^x = x \iff x \ln(x) = \ln(x) \iff \ln(x)(x - 1) = 0 \iff x = 1.
\]

Step 2. Identify the location of the initial term relative to the fixed points and the domain intervals of \( f \):
We continue to analyze the sequence further on (0, 1).

Step 3. \( f \) maps (0, 1) into itself: We have

\[
0 < f(x) = x^x = e^{x \ln(x)} < e^0 = 1 \text{ for } 0 < x < 1.
\]

Note that \( \ln(x) < 0 \) for \( 0 < x < 1 \), so \( x \ln(x) < 0 \) for \( 0 < x < 1 \).

Step 4. From the previous steps we can conclude that the sequence \( \{a_n\} \) is monotonic and stays inside the interval (0, 1). Because \( a_2 = 1/\sqrt{2} > 1/2 = a_1 \), the sequence is increasing, and we have \( \lim_{n \to \infty} a_n = 1 \).

An important component of the above procedure to analyze recursive sequences for eventual monotonicity is to verify whether a continuous function \( f \) maps an interval \((a, b)\), \(-\infty \leq a < b \leq \infty\), into itself, see Steps 3. and 4. This really is the question about how large or small the values \( f(x) \) can become as \( x \) varies over \((a, b)\). Note that if \( a \) and \( b \) are both finite and \( f \) extends continuously to \([a, b]\), then answering this question is just what the Closed Interval Method accomplishes. In the general case we can still proceed very similar to the Closed Interval Method: If all the local minima and maxima of \( f \) as \( x \) varies over \((a, b)\) fall inside the range \((a, b)\), and if the limits of \( f \) towards both interval ends exist and neither “overshoot” the value \( b \) nor “undershoot” the value \( a \), then \( f \) maps \((a, b)\) into itself. This leads to the following:

**Strategy for checking whether \( f \) maps \((a, b)\) into itself**

We assume that \( f \) is continuous on \((a, b)\) and that both \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to b^-} f(x) \) exist, where each is either finite or \( \pm \infty \).

A. Find all critical points of \( f \) in \((a, b)\).

B. Find \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to b^-} f(x) \).

Note that if \( f \) extends continuously to \( a \) or \( b \) (if finite), the corresponding endpoint limit will just be \( f(a) \) or \( f(b) \), respectively.

C. Tabulate the function values of \( f \) at the critical points, and both limits. If the values at all the critical points fall inside \((a, b)\), and the limits are both \( \geq a \) and \( \leq b \), then \( f \) maps \((a, b)\) into itself. Otherwise it does not.

A *Special Case*: If \( f \) is increasing on \((a, b)\) and extends continuously to the finite endpoints of \((a, b)\) (if any) with \( f(a) \geq a \) as well as \( f(b) \leq b \), respectively, then \( f \) maps \((a, b)\) into itself. Note that if \( a \) or \( b \) are fixed points, then the inequalities pertaining to the function values are satisfied. Recall that to check whether \( f \) is increasing on \((a, b)\) it is sufficient to verify that \( f'(x) > 0 \) for all \( a < x < b \).

**Example 2.** Consider the recursive sequence

\[
\begin{align*}
a_{n+1} &= \frac{5}{6 - a_n}, \
a_1 &= 4,
\end{align*}
\]

Show that \( \{a_n\} \) converges, and find \( \lim_{n \to \infty} a_n \).
Solution: Consider the function \( f(x) = \frac{5}{6-x} \). Observe that \( a_{n+1} = f(a_n) \) for all \( n = 1, 2, 3, \ldots \)

Step 1. Solve the fixed point equation:

\[
\frac{5}{6-x} = x \iff 6x - x^2 = 5 \iff x^2 - 6x + 5 = 0 \iff x = 1 \text{ or } x = 5.
\]

Step 2. Identify the location of the initial term relative to the fixed points and the domain intervals of \( f \):

We thus analyze the sequence further on the interval \((1, 5)\).

Step 3. \( f \) maps the interval \((1, 5)\) to itself:

Because both 1 and 5 are fixed points, it is enough to show that \( f \) is increasing on \((1, 5)\). But this is obvious in view of \( f'(x) = \frac{5}{(6-x)^2} > 0 \) for all \( x \neq 6 \).

Step 4. Based on the previous three steps, we conclude that the sequence \( \{a_n\} \) is monotonic and stays in the interval \((1, 5)\). Since \( a_2 = \frac{3}{2} < 4 = a_1 \) the sequence is decreasing, and we have \( \lim_{n \to \infty} a_n = 1 \).

Example 3. Consider the recursive sequence

\[
\begin{cases}
  a_{n+1} = \frac{a_n^2 - 2}{2a_n - 3}, & n = 1, 2, 3, \ldots, \\
  a_1 = \text{Any value other than } \frac{3}{2}.
\end{cases}
\]

Show that \( \{a_n\} \) converges, and find \( \lim_{n \to \infty} a_n \).

Solution: Consider the function \( f(x) = \frac{x^2 - 2}{2x - 3} \). Then \( a_{n+1} = f(a_n) \) for all \( n = 1, 2, 3, \ldots \)

Step 1. Find the fixed points:

\[
\frac{x^2 - 2}{2x - 3} = x \iff x^2 - 2 = 2x^2 - 3x \iff x^2 - 3x + 2 = 0 \iff x = 1 \text{ or } x = 2.
\]

Step 2. The further analysis will depend on the location of \( a_1 \) relative to the marked points on the real number line shown:

We proceed with the process, being mindful that we have to distinguish several cases.

Step 3. We begin by analyzing where \( f \) increases and decreases:

\[
f'(x) = \frac{(2x - 3)2x - (x^2 - 2)2}{(2x - 3)^2} = \frac{2x^2 - 6x + 4}{(2x - 3)^2} = \frac{2(x-1)(x-2)}{(2x-3)^2}.
\]

From the formula we deduce that \( f'(x) > 0 \) for \(-\infty < x < 1 \) and \( 2 < x < \infty \). Since both \( x = 1 \) and \( x = 2 \) are fixed points, we get that \( f \) maps \((-\infty, 1)\) into itself, and that \( f \) maps \((2, \infty)\) into itself.

Because \( f'(x) < 0 \) for \( 1 < x < \frac{3}{2} \) and \( \frac{3}{2} < x < 2 \), we deduce that \( f \) is decreasing on \([1, \frac{3}{2}]\) and on \((\frac{3}{2}, 2]\). Consequently, \( 1 = f(1) > f(x) \) for all \( 1 < x < \frac{3}{2} \), which shows that \( f \) maps \((1, \frac{3}{2})\) into \((-\infty, 1)\). Likewise,
Step 4. From the previous steps, we can infer the following:

- If \(-\infty < a_1 < 1\), the sequence \(\{a_n\}\) is monotonic and stays inside the interval \((−\infty, 1)\). Because \(f(0) = 2/3 > 0\) we have that \(f(x) > x\) on \((−\infty, 1)\), and consequently the sequence \(\{a_n\}\) is increasing with \(\lim_{n→∞} a_n = 1\).
- If \(2 < a_1 < ∞\), the sequence \(\{a_n\}\) is monotonic and stays inside the interval \((2, ∞)\). Because \(f(3) = 7/3 < 3\) we have that \(f(x) < x\) on \((2, ∞)\), and consequently the sequence \(\{a_n\}\) is decreasing with \(\lim_{n→∞} a_n = 2\).
- If \(a_1 = 1\) then \(a_n = 1\) for all \(n\), and consequently \(\lim_{n→∞} a_n = 1\). Likewise, if \(a_1 = 2\) then \(a_n = 2\) for all \(n\), and thus \(\lim_{n→∞} a_n = 2\).
- If \(1 < a_1 < 3/2\), then \(a_2 = f(a_1)\) will be in the interval \((−\infty, 1)\). Consequently, starting from the second sequence term \(a_2\), the sequence will increase with \(\lim_{n→∞} a_n = 1\).
- If \(3/2 < a_1 < 2\), then \(a_2 = f(a_1)\) will be in the interval \((2, ∞)\). Thus, starting from the second sequence term, the sequence decreases with \(\lim_{n→∞} a_n = 2\).

In conclusion:

\[
\lim_{n→∞} a_n = \begin{cases} 
1 & \text{if } a_1 < \frac{3}{2}, \\
2 & \text{if } a_1 > \frac{3}{2}.
\end{cases}
\]

Example 4. The Fibonacci numbers are the sequence \(\{F_n\}_{n=0}^{∞}\) defined as

\[
\begin{align*}
F_{n+1} &= F_n + F_{n-1}, & n = 1, 2, 3, \ldots, \\
F_0 &= 0, \\
F_1 &= 1.
\end{align*}
\]

The sequence \(\{a_n\}_{n=1}^{∞}\) of ratios of subsequent Fibonacci numbers is given by

\[
a_n = \frac{F_{n+1}}{F_n}, \quad n = 1, 2, 3, \ldots.
\]

(a) Show that \(\{a_n\}_{n=1}^{∞}\) is a recursive sequence. Determine explicitly a function \(g\) such that \(a_{n+1} = g(a_n)\) for all \(n = 1, 2, 3, \ldots\)

(b) Find \(a_1, \ldots, a_{10}\). Is the sequence \(\{a_n\}\) monotonic? What if we consider separately \(a_1, a_3, a_5, a_7, a_9\) (just the odd terms) or \(a_2, a_4, a_6, a_8, a_{10}\) (just the even terms)?

(c) Define \(c_n = a_{2n-1}\) and \(d_n = a_{2n}\), \(n = 1, 2, 3, \ldots\) Then \(\{c_n\}\) and \(\{d_n\}\) are subsequences that consist of the odd- and even-indexed terms of \(\{a_n\}\), respectively. Show that both \(\{c_n\}\) and \(\{d_n\}\) are recursive sequences by determining explicitly a function \(f\) with \(c_{n+1} = f(c_n)\) and \(d_{n+1} = f(d_n)\) for all \(n = 1, 2, 3, \ldots\)

(d) Show that both \(\{c_n\}\) and \(\{d_n\}\) converge to the same limit \(\varphi\).

Since the odd- and even-indexed subsequences of \(\{a_n\}\) both converge to the same limit \(\varphi\), we conclude that \(\{a_n\}\) also converges with \(\lim_{n→∞} a_n = \varphi\). The limit \(\varphi\), which we will find in the solution to be \(\frac{1+\sqrt{5}}{2}\), is known as the Golden Ratio or the Golden Mean.
Solution:

(a) Using the recurrence relation for the Fibonacci numbers we obtain

\[ a_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{a_{n-1}}. \]

Note that \( a_1 = \frac{F_2}{F_1} = \frac{F_1 + F_0}{F_1} = 1 \). This leads to

\[
\begin{align*}
\begin{cases}
  a_{n+1} = 1 + \frac{1}{a_n}, & n = 1, 2, 3, \ldots, \\
  a_1 = 1.
\end{cases}
\end{align*}
\]

The function \( g \) is thus given by \( g(x) = 1 + \frac{1}{x} \).

(b) The first 10 sequence terms are shown in the table below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/1 = 1</td>
</tr>
<tr>
<td>2</td>
<td>2/1 = 2</td>
</tr>
<tr>
<td>3</td>
<td>3/2 = 1.5</td>
</tr>
<tr>
<td>4</td>
<td>5/3 = 1.6667</td>
</tr>
<tr>
<td>5</td>
<td>8/5 = 1.6</td>
</tr>
<tr>
<td>6</td>
<td>13/8 = 1.625</td>
</tr>
<tr>
<td>7</td>
<td>21/13 = 1.615384</td>
</tr>
<tr>
<td>8</td>
<td>34/21 = 1.619047</td>
</tr>
<tr>
<td>9</td>
<td>55/34 = 1.61764705882352941</td>
</tr>
<tr>
<td>10</td>
<td>89/55 = 1.618</td>
</tr>
</tbody>
</table>

The sequence \( \{a_n\} \) is not monotonic. Instead, it seems to alternate: The table suggests that the successor to any odd-indexed term may be greater, while the successor to any even-indexed term may be smaller. Moreover, from just tracking the even- and odd-indexed terms separately, it appears that the even-indexed terms decrease, while the odd-indexed terms appear to increase. We shall see below that this is indeed the case.

(c) We have:

\[
\begin{align*}
  c_1 &= a_1 & d_1 &= a_2 = g(a_1) \\
  c_2 &= a_3 = g(a_2) = (g \circ g)(c_1) & d_2 &= a_4 = g(a_3) = (g \circ g)(d_1) \\
  c_3 &= a_5 = g(a_4) = (g \circ g)(c_2) & d_3 &= a_6 = g(a_5) = (g \circ g)(d_2) \\
  c_4 &= a_7 = (g \circ g)(c_3) & d_4 &= a_8 = (g \circ g)(d_3) \\
  c_5 &= a_9 = (g \circ g)(c_4) & d_5 &= a_{10} = (g \circ g)(d_4) \\
  c_{n+1} &= (g \circ g)(c_n) & d_{n+1} &= (g \circ g)(d_n) \\
  c_1 &= a_1 & d_1 &= a_2
\end{align*}
\]

We see that \( \{c_n\} \) and \( \{d_n\} \) are recursive sequences with the function \( g \circ g \). Observe that this is generally true and not limited to the particular sequence \( \{a_n\} \) considered in this example.

In the specific case considered here we have

\[
(g \circ g)(x) = 1 + \frac{1}{\frac{2x+1}{x+1}} = \frac{2x+1}{x+1},
\]
so \( f(x) = \frac{2x+1}{x+1} \). We have

\[
\begin{align*}
  c_{n+1} &= \frac{2c_n + 1}{c_n + 1}, \quad n = 1, 2, 3, \ldots, \\
  d_{n+1} &= \frac{2d_n + 1}{d_n + 1}, \quad n = 1, 2, 3, \ldots, \\
  c_1 &= 1 \\
  d_1 &= 2
\end{align*}
\]

(d) We analyze \( \{c_n\} \) and \( \{d_n\} \) analogously to the previous examples.

Step 1. Find the fixed points of \( f \):

\[
\frac{2x + 1}{x + 1} = x \iff 2x + 1 = x^2 + x \iff x^2 - x - 1 = 0 \iff x = \frac{1 \pm \sqrt{5}}{2}.
\]

Step 2. Locate \( c_1 \) and \( d_1 \) relative to the fixed points and the domain intervals of \( f \):

We consequently proceed to analyze \( \{c_n\} \) further on \( \left(\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right) \), and \( \{d_n\} \) on \( \left(\frac{1 + \sqrt{5}}{2}, \infty\right) \).

Step 3. \( f \) maps both intervals \( \left(\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right) \) and \( \left(\frac{1 + \sqrt{5}}{2}, \infty\right) \) into themselves:

Since both \( \frac{1 \pm \sqrt{5}}{2} \) are fixed points, it is enough to show that \( f \) is increasing on both intervals. But since \( f'(x) = \frac{1}{(x+1)^2} > 0 \) for all \( x \neq -1 \) this is clear.

Step 4. Based on the previous steps we conclude that \( \{c_n\} \) is increasing, stays in the interval \( \left(\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\right) \), and \( \lim_{n \to \infty} c_n = \frac{1 + \sqrt{5}}{2} \). Likewise, \( \{d_n\} \) is decreasing, stays in the interval \( \left(\frac{1 + \sqrt{5}}{2}, \infty\right) \), and \( \lim_{n \to \infty} d_n = \frac{1 + \sqrt{5}}{2} \).

The following cobweb diagrams illustrate the situation.
4. Problems

Draw a cobweb diagram for the recursive sequence defined by $a_{n+1} = f(a_n)$, where the graph of $f$ is shown and $a_1$ is marked on the $x$-axis. Show at least $a_2, \ldots, a_7$.

Show that the following sequences converge, and find the limit.

3. \[ \begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 8), \quad n \geq 1, \\
&= 4 \\
a_{n+1} &= a_n^2 - 2a_n + 2, \quad n \geq 1, \\
&= \frac{3}{2} \\
a_{n+1} &= \sqrt{3a_n}, \quad n \geq 1, \\
&= 1 \\
a_{n+1} &= -\sqrt{24 - 2a_n}, \quad n \geq 1, \\
&= 11 \\
a_{n+1} &= 2a_n^{2/3}, \quad n \geq 1, \\
&= 1 \\
a_{n+1} &= 2 - \frac{1}{2 + a_n}, \quad n \geq 1, \\
&= 5 \\
a_{n+1} &= \frac{5a_n}{3 + a_n}, \quad n \geq 1, \\
&= 1 \\
\end{aligned} \]

4. \[ \begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 9), \quad n \geq 1, \\
&= 15 \\
a_{n+1} &= 1 + \ln(a_n), \quad n \geq 1, \\
&= 5 \\
a_{n+1} &= a_n + \frac{\sin(a_n)}{e^{a_n}}, \quad n \geq 1, \\
&= \frac{3\pi}{2} \\
a_{n+1} &= \tan(a_n), \quad n \geq 1, \\
&= \frac{\pi}{4} \\
a_{n+1} &= a_n - \frac{\tan(a_n) - 1}{1 + \tan^2(a_n)}, \quad n \geq 1, \\
&= \frac{\pi}{3} \\
a_{n+1} &= a_n \left(\frac{1}{2} - \ln(a_n)\right), \quad n \geq 1, \\
&= 1 \\
\end{aligned} \]

5. Consider the recursive sequence
\[ a_{n+1} = \frac{4 \arctan(a_n)}{\pi}, \quad n \geq 1, \]
where $a_1$ is arbitrary. Show that $\lim_{n \to \infty} a_n$ exists and find it.

6. Consider the recursive sequence
\[ a_{n+1} = \frac{3a_n - a_n^3}{2}, \quad n \geq 1, \]
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where $-2 < a_1 < 2$ is arbitrary. Show that $\lim_{n \to \infty} a_n$ exists and find it.

Show that the function $f$ maps the interval $I$ into itself.

18. $f(x) = 3.9x(1 - x)$, $I = (0, 1)$.

19. $f(x) = \frac{3x(1 - 2\ln(x))}{4}$, $I = (0, \sqrt{e})$.

Use the approach from Example 4(c),(d) to show that the following sequences converge, and find the limit.

20. \[ \begin{cases} a_{n+1} = \frac{18 - a_n}{2}, & n \geq 1, \\ a_1 = 4 \end{cases} \]

21. \[ \begin{cases} a_{n+1} = 2 + \frac{1}{2 + a_n}, & n \geq 1, \\ a_1 = 5 \end{cases} \]

22. Show that for every $a_1$ the recursive sequence defined by $a_{n+1} = \cos(a_n)$, $n \geq 1$, converges to the unique fixed point of the cosine function.

23. Repelling fixed points: Let $f$ be differentiable, and let $f(a) = a$.
   (a) Let $f'(a) > 1$. Show that there is $h_0 > 0$ such that $f(x) > x$ for $a < x < a + h_0$, and $f(x) < x$ for $a - h_0 < x < a$. Deduce that if $\{a_n\}$ is recursive with $a_{n+1} = f(a_n)$, and if $\lim_{n \to \infty} a_n = a$, then necessarily $a_n = a$ for all $n$ large enough.

   Hint: Consider $g(x) = f(x) - x$. Then $g(a) = 0$ and $g'(a) > 0$ (why?). Then use the limit definition of the derivative to derive the stated inequalities.

   Finally, note that if $a < a_n < a + h_0$, then $a_{n+1} < a_n$, and if $a - h_0 < a_n < a$ then $a_{n+1} < a_n$, so the sequence moves away from $a$.

   (b) Now let $f'(a) < -1$. Use the first part and the approach from Example 4 to show that if $\{a_n\}$ is recursive with $a_{n+1} = f(a_n)$, and if $\lim_{n \to \infty} a_n = a$,

   then $a_n = a$ for all $n$ large enough.

24. Let $f$ be twice differentiable. Let $f(a) = 0$. Suppose that $f'(x) \neq 0$ for $a < x < b$, and $f(x) \cdot f''(x) > 0$ for all $a < x < b$. Consider the sequence $\{a_n\}$ defined by Newton’s method

   \[ \begin{cases} a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}, & n \geq 1, \\ a_1 = \text{Any value in } (a, b). \end{cases} \]

   Show that $\{a_n\}$ stays in $(a, b)$, and is decreasing with $\lim_{n \to \infty} a_n = a$. 

Appendix A. Rigorous Proofs

Theorem 1. Let \( I = (a, b) \) be an open interval, where \(-\infty \leq a < b \leq \infty\), and let \( f : I \to \mathbb{R} \) be continuous such that \( f(x) \neq x \) for all \( x \in I \). Then either
(A) \( f(x) > x \) for all \( x \in I \), or
(B) \( f(x) < x \) for all \( x \in I \).

Now suppose further that \( f(I) \subset I \), and consider the recursive sequence
\[
\begin{align*}
a_{n+1} &= f(a_n), \quad n = 1, 2, \ldots, \\
a_1 &= \text{arbitrary initial value in } I.
\end{align*}
\]

This sequence is well-defined and contained in the interval \( I \), and in Case (A) it is increasing with \( \lim_{n \to \infty} a_n = b \), and in Case (B) it is decreasing with \( \lim_{n \to \infty} a_n = a \).

Proof. Consider the function \( g : I \to \mathbb{R} \) defined by \( g(x) = f(x) - x \). Then \( g \) is continuous, and by assumption we have \( g(x) \neq 0 \) for all \( x \in I \). Therefore, by the Intermediate Value Theorem, we have either \( g(x) > 0 \) for all \( x \in I \) which leads to Case (A), or \( g(x) < 0 \) for all \( x \in I \) which leads to Case (B).

Next consider the sequence \( \{a_n\}_{n=1}^{\infty} \): We will first show inductively that \( a_n \in I \) for all \( n \in \mathbb{N} \), and then that \( \{a_n\}_{n=1}^{\infty} \) is either increasing or decreasing depending on whether Case (A) or Case (B) holds for the function \( f \).

By assumption, the initial value \( a_1 \) belongs to \( I \). So suppose that we know that \( a_n \in I \) for some \( n \in \mathbb{N} \). Because \( f(I) \subset I \) we obtain that \( a_{n+1} = f(a_n) \in I \). This completes the induction and shows that the sequence \( \{a_n\}_{n=1}^{\infty} \) is well-defined and contained in \( I \). Moreover, for all \( n \in \mathbb{N} \) we have \( a_{n+1} = f(a_n) > a_n \) in Case (A), and \( a_{n+1} = f(a_n) < a_n \) in Case (B), proving the claim about the monotonicity of the sequence \( \{a_n\}_{n=1}^{\infty} \).

It remains to prove the statement about the limit. Suppose that Case (A) holds for the function \( f \), so the sequence \( \{a_n\}_{n=1}^{\infty} \) is increasing. By the Monotonic Sequence Theorem, we have that
\[
\lim_{n \to \infty} a_n = L = \sup \{a_n \mid n \in \mathbb{N}\},
\]
which is either finite or infinite. Because \( a_n < b \) for all \( n \in \mathbb{N} \), we obtain that necessarily \( L \leq b \). The assumption that \( L < b \) leads to a contradiction, as follows: As \( a_1 \leq L < b \), we necessarily have \( L \in I \), and by continuity of the function \( f \) at the point \( L \in I \) we get
\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} f(a_n) = f(L).
\]
By our stated assumption \( f \) does not have any fixed points in \( I \), so this is impossible. Consequently, \( b = L \). The argument in Case (B) is the same, thus concluding the proof of the theorem. \( \square \)

Remark 1. Note that neither the statement of Theorem 1 nor its proof make any assumptions nor conclusions about the monotonicity of the function \( f : I \to \mathbb{R} \) itself. The monotonicity assertion of the theorem exclusively pertains to the sequence \( \{a_n\}_{n=1}^{\infty} \). To clarify this further by means of an example consider the function
\[
f(x) = \frac{x}{2} - x^2 \sin \left( \frac{1}{x} \right)
\]
on the interval \( I = (0, \frac{1}{2}) \). \( f \) is continuous (even \( C^\infty \)) on \( I \). Let us verify that \( 0 < f(x) < x \) for all \( x \in I \), i.e., the function \( f \) satisfies the assumptions of Theorem 1
on the interval $I$, and Case (B) of the theorem applies: Because $0 < x < \frac{1}{2}$ we see that

$$f(x) = 0 \iff \sin\left(\frac{1}{x}\right) = \frac{1}{2x} > 1,$$

and thus there are no solutions to the equation $f(x) = 0$ in $I$. As $\frac{1}{2} \in I$ with $f(\frac{1}{2}) = \frac{1}{2\pi} > 0$ we necessarily obtain that $f(x) > 0$ on $I$ by the Intermediate Value Theorem. Analogously, using again that $0 < x < \frac{1}{2}$, we obtain that

$$f(x) = x \iff \sin\left(\frac{1}{x}\right) = -\frac{1}{2x} < -1,$$

and consequently the equation $f(x) = x$ has no solutions in $I$ either. Because $f(\frac{1}{2}) = \frac{1}{2\pi} < \frac{1}{2}$ we get that $f(x) < x$ on $I$ from the Intermediate Value Theorem.

Theorem 1 now implies that every recursive sequence $a_{n+1} = f(a_n)$ with arbitrary initial value $0 < a_1 < \frac{1}{2}$ is decreasing with $\lim_{n \to \infty} a_n = 0$. The function $f$ itself, however, is not monotonic in any neighborhood of the origin. This follows because

$$f'(x) = \frac{1}{2} - 2x \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

changes sign infinitely many times on every interval of the form $(0, \varepsilon)$ for every $0 < \varepsilon < \frac{1}{2}$ as can be seen, for example, from the fact that $f'(\frac{1}{k\pi}) = \frac{1}{2} + (-1)^k$ for all $k \in \mathbb{N}$. For an illustration of this example see Figure 6.

![Figure 6. An illustration of the example discussed in Remark 1.](image)

**Theorem 2.** Let $(a, b)$ be an open interval, where $-\infty \leq a < b \leq \infty$, and let $f : (a, b) \to \mathbb{R}$ be continuous. Let $-\infty \leq \alpha < \beta \leq \infty$. Then the following are equivalent:

1. $f$ maps $(a, b)$ into $(\alpha, \beta)$.
2. For all critical points $\xi$ of $f$ in $(a, b)$ we have $\alpha < f(\xi) < \beta$. Moreover, $\alpha \leq \limsup_{x \to a^+} f(x)$ and $\liminf_{x \to a^+} f(x) \leq \beta$, as well as $\alpha \leq \limsup_{x \to b^-} f(x)$ and $\liminf_{x \to b^-} f(x) \leq \beta$.

**Proof.** If $f$ maps $(a, b)$ into $(\alpha, \beta)$, all the claimed properties about the function values at critical points and the limits towards the interval ends are trivial. Hence suppose that $\alpha < f(\xi) < \beta$ for all critical points $\xi \in (a, b)$, and that $\alpha \leq \limsup_{x \to a^+} f(x)$
and \( \liminf_{x \to a^+} f(x) \leq \beta \), as well as \( \alpha \leq \limsup_{x \to b^-} f(x) \) and \( \liminf_{x \to b^-} f(x) \leq \beta \), hold. We have to show that \( f \) maps \((a, b)\) into \((\alpha, \beta)\).

Assume that this was not the case. Then there exists \( x_0 \in (a, b) \) such that either \( f(x_0) \geq \beta \) or \( f(x_0) \leq \alpha \). Without loss of generality, we consider only the case that \( f(x_0) \geq \beta \) since the argument in the other situation is analogous. We claim that there must exist \( a < c < x_0 \) and \( x_0 < d < b \) such that \( f(c) < f(x_0) \) and \( f(d) < f(x_0) \). Once this is warranted we obtain, using the Extreme Value Theorem, that \( M = \max \{ f(x) \mid c \leq x \leq d \} \geq f(x_0) \geq \beta \), and there must exist \( c < \xi_0 < d \) with \( f(\xi_0) = M \). But this implies that \( f \) attains a local maximum at \( \xi_0 \), and consequently \( \xi_0 \) must be a critical point for \( f \). By our present assumption we therefore have \( f(\xi_0) = M < \beta \), which contradicts with \( M \geq \beta \).

We need to establish the claim about the existence of \( c \) and \( d \) with the stated properties. If there exists a critical point of \( f \) in \((a, x_0)\), choose \( c \) to be such a critical point. By our present assumption on the function values at critical points, we then have \( f(c) < \beta < f(x_0) \) as desired. If there is no critical point of \( f \) in \((a, x_0)\), we have that \( f' \) exists everywhere and vanishes nowhere on \((a, x_0)\). By the Darboux Theorem, \( f' \) has the intermediate value property, so either \( f' > 0 \) or \( f' < 0 \) everywhere on \((a, x_0)\). The case that \( f' < 0 \) is impossible as this would imply that \( f \) strictly decreases on \((a, x_0] \), which would entail that \( \lim_{x \to a^+} f(x) \) exists with \( \lim_{x \to a^+} f(x) > f(x_0) \geq \beta \) contrary to our present assumption on that limit. Consequently, \( f' > 0 \) on \((a, x_0)\), which shows that \( f \) is strictly increasing on \((a, x_0)\]. Hence we can pick \( c \) arbitrarily with \( a < c < x_0 \) and have \( f(c) < f(x_0) \) as desired. The existence of the point \( d \) with \( x_0 < d < b \) and \( f(d) < f(x_0) \) follows in the same manner. \( \square \)

References