Reduction of Order

Given a linear second order ordinary differential equation of the form

\[ y'' + p(x)y' + q(x)y = f(x). \]  \hfill (1)

The task is to find solutions \( y = y(x) \) to (1), given a solution \( y_1 = y_1(x) \) to the homogeneous equation

\[ y'' + p(x)y' + q(x)y = 0. \]

The idea is to find solutions in the form \( y(x) = v(x)y_1(x) \) with a function \( v = v(x) \) that is to be determined, as follows. According to the product rule we have

\[
\begin{align*}
y(x) &= v(x)y_1(x), \\
y'(x) &= v'(x)y_1(x) + v(x)y_1'(x), \\
y''(x) &= v''(x)y_1(x) + 2v'(x)y_1'(x) + v(x)y_1''(x).
\end{align*}
\]

With this setup for \( y(x) \) we thus get

\[
y'' + p(x)y' + q(x)y = y_1(x)v''(x) + (2y_1'(x) + p(x)y_1(x))v'(x) + (y_1''(x) + p(x)y_1'(x) + q(x)y_1(x))v(x) \\
= y_1(x)v''(x) + (2y_1'(x) + p(x)y_1(x))v'(x).
\]

Note that \( y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0 \) by assumption on \( y_1 \). Consequently, for \( y(x) = v(x)y_1(x) \) to solve (1), we must have

\[
y_1(x)v''(x) + (2y_1'(x) + p(x)y_1(x))v'(x) = f(x). \hfill (2)
\]

Equation (2) is a linear differential equation for \( w(x) = v'(x) \) of first order! We can solve this to first obtain \( w = v' \), and integrate the solution one more time to find the function \( v = v(x) \). In this way, we find solutions \( y(x) = v(x)y_1(x) \) to (1).

Examples

1. Find the general solution to

\[
xy'' - (2x + 1)y' + (x + 1)y = x^2, \quad x > 0,
\]

given that \( y_1(x) = e^x \) is a solution to the homogeneous equation.

Solution: Rewrite the equation as

\[
y'' - \left(2 + \frac{1}{x}\right)y' + \left(1 + \frac{1}{x}\right)y = x.
\]

We seek \( y \) in the form \( y(x) = v(x)y_1(x) \) where \( v(x) \) solves equation (2), i.e.,

\[
e^x v''(x) + \left(2e^x - \left(2 + \frac{1}{x}\right)e^x \right)v'(x) = x.
\]

We divide by the leading term \( e^x \) and get with \( w = v' \)

\[
w'(x) - \frac{1}{x}w(x) = xe^{-x}.
\]

An integrating factor is \( e^{-\int \frac{1}{x} \, dx} = x^{-1} \), and consequently \( (x^{-1}w)' = e^{-x} \). This shows that

\[
v'(x) = w(x) = -xe^{-x} + Cx,
\]

and consequently

\[
v(x) = C \frac{x^2}{2} + (x + 1)e^{-x} + D
\]
The solutions $y$ are therefore given by

$$y(x) = v(x)y_1(x) = \frac{C}{2}x^2e^x + De^x + x + 1$$  \hspace{1cm} (3)$$

with arbitrary constants $C$ and $D$.

**Note:** From the form of (3) and the structural results about linear differential equations of second order we see that a fundamental set of the differential equation is given by

$$y_1(x) = e^x \quad \text{and} \quad y_2(x) = x^2e^x.$$  

2. Find the general solution to

$$x^2y'' + xy' - y = x^2 + 1, \quad x > 0,$$

given that $y_1(x) = x$ is a solution to the homogeneous equation.

**Solution:** Rewrite the equation as

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 1 + \frac{1}{x^2}.$$  

We seek $y$ in the form $y(x) = v(x)y_1(x)$ where $v(x)$ solves equation (2), i.e.,

$$xv''(x) + \left(2 + \frac{1}{x}\right)v'(x) = 1 + x^{-2}.$$  

We divide by the leading term $x$ and get with $w = v'$

$$w'(x) + \frac{3}{x}w(x) = x^{-1} + x^{-3}.$$  

An integrating factor is $e^{\int \frac{3}{x}dx} = x^3$, and consequently $(x^3w)' = x^2 + 1$. This shows that

$$v'(x) = w(x) = \frac{1}{3}x^{-1} + Cx^{-3}$$  

and consequently

$$v(x) = \frac{1}{3}x - x^{-1} - Cx^{-2} + D.$$  

The solutions $y$ are therefore given by

$$y(x) = v(x)y_1(x) = \frac{1}{3}x^2 - 1 - \frac{C}{2}x^{-1} + Dx$$  \hspace{1cm} (4)$$

with arbitrary constants $C$ and $D$.

**Note:** As in the previous example, we infer from (4) that a fundamental set of the differential equation is given by

$$y_1(x) = x \quad \text{and} \quad y_2(x) = x^{-1}.$$