36. Consider the autonomous DE $dy/dx = y^2 - y - 6$. Use your ideas from Problem 35 to find intervals on the $y$-axis for which solution curves are concave up and intervals for which solution curves are concave down. Discuss why each solution curve of an initial-value problem of the form $dy/dx = y^2 - y - 6$, $y(0) = y_0$, where $-2 < y_0 < 3$, has a point of inflection with the same $y$-coordinate. What is that $y$-coordinate? Carefully sketch the solution curve for which $y(0) = -1$. Repeat for $y(2) = 2$.

37. Suppose the autonomous DE in (1) has no critical points. Discuss the behavior of the solutions.

### Mathematical Models

38. **Population Model** The differential equation in Example 3 is a well-known population model. Suppose the DE is changed to

$$\frac{dP}{dt} = P(aP - b),$$

where $a$ and $b$ are positive constants. Discuss what happens to the population $P$ as time $t$ increases.

39. **Population Model** Another population model is given by

$$\frac{dP}{dt} = kP - h,$$

where $h$ and $k$ are positive constants. For what initial values $P(0) = P_0$ does this model predict that the population will go extinct?

40. **Terminal Velocity** In Section 1.3 we saw that the autonomous differential equation

$$m \frac{dv}{dt} = mg - kv,$$

where $k$ is a positive constant and $g$ is the acceleration due to gravity, is a model for the velocity $v$ of a body of mass $m$ that is falling under the influence of gravity. Because the term $-kv$ represents air resistance, the velocity of a body falling from a great height does not increase without bound as time $t$ increases. Use a phase portrait of the differential equation to find the limiting, or terminal, velocity of the body. Explain your reasoning.

41. Suppose the model in Problem 40 is modified so that air resistance is proportional to $v^2$, that is,

$$m \frac{dv}{dt} = mg - kv^2.$$

See Problem 17 in Exercises 1.3. Use a phase portrait to find the terminal velocity of the body. Explain your reasoning.

42. **Chemical Reactions** When certain kinds of chemicals are combined, the rate at which the new compound is formed is modeled by the autonomous differential equation

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X),$$

where $k > 0$ is a constant of proportionality and $\beta > \alpha > 0$. Here $X(t)$ denotes the number of grams of the new compound formed in time $t$.

(a) Use a phase portrait of the differential equation to predict the behavior of $X(t)$ as $t \to \infty$.

(b) Consider the case when $\alpha = \beta$. Use a phase portrait of the differential equation to predict the behavior of $X(t)$ as $t \to \infty$ when $X(0) < \alpha$. When $X(0) > \alpha$.

(c) Verify that an explicit solution of the DE in the case when $k = 1$ and $\alpha = \beta$ is $X(t) = \alpha - 1/(t + c)$. Find a solution that satisfies $X(0) = \alpha/2$. Then find a solution that satisfies $X(0) = 2\alpha$. Graph these two solutions. Does the behavior of the solutions as $t \to \infty$ agree with your answers to part (b)?

### 2.2 SEPARABLE VARIABLES

#### REVIEW MATERIAL
- Basic integration formulas (See inside front cover)
- Techniques of integration: integration by parts and partial fraction decomposition
- See also the Student Resource and Solutions Manual.

#### INTRODUCTION
We begin our study of how to solve differential equations with the simplest of all differential equations: first-order equations with separable variables. Because the method in this section and many techniques for solving differential equations involve integration, you are urged to refresh your memory on important formulas (such as $\int du/u$) and techniques (such as integration by parts) by consulting a calculus text.
Consider the first-order differential equation \( \frac{dy}{dx} = f(x, y) \). When \( f \) does not depend on the variable \( y \), that is, \( f(x, y) = g(x) \), the differential equation

\[
\frac{dy}{dx} = g(x) \tag{1}
\]

can be solved by integration. If \( g(x) \) is a continuous function, then integrating both sides of (1) gives \( y = \int g(x) \, dx = G(x) + c \), where \( G(x) \) is an antiderivative (indefinite integral) of \( g(x) \). For example, if \( \frac{dy}{dx} = 1 + e^{2x} \), then its solution is \( y = x + \frac{1}{2}e^{2x} + c \).

A DEFINITION  Equation (1), as well as its method of solution, is just a special case when the function \( f \) in the normal form \( \frac{dy}{dx} = f(x, y) \) can be factored into a function of \( x \) times a function of \( y \).

DEFINITION 2.2.1  Separable Equation

A first-order differential equation of the form

\[
\frac{dy}{dx} = g(x)h(y) \tag{2}
\]

is said to be separable or to have separable variables.

For example, the equations

\[
\frac{dy}{dx} = y^2xe^{3x+4y} \quad \text{and} \quad \frac{dy}{dx} = y + \sin x
\]

are separable and nonseparable, respectively. In the first equation we can factor \( f(x, y) = y^2xe^{3x+4y} \) as

\[
f(x, y) = y^2xe^{3x+4y} = (xe^{3x})(y^2e^{4y}),
\]

but in the second equation there is no way of expressing \( y + \sin x \) as a product of a function of \( x \) times a function of \( y \).

Observe that by dividing by the function \( h(y) \), we can write a separable equation \( \frac{dy}{dx} = g(x)h(y) \) as

\[
p(y) \frac{dy}{dx} = g(x), \tag{2}
\]

where, for convenience, we have denoted \( 1/h(y) \) by \( p(y) \). From this last form we can see immediately that (2) reduces to (1) when \( h(y) = 1 \).

Now if \( y = \phi(x) \) represents a solution of (2), we must have \( p(\phi(x))\phi'(x) = g(x) \), and therefore

\[
\int p(\phi(x))\phi'(x) \, dx = \int g(x) \, dx \tag{3}
\]

But \( dy = \phi'(x) \, dx \), and so (3) is the same as

\[
\int p(y) \, dy = \int g(x) \, dx \quad \text{or} \quad H(y) = G(x) + c, \tag{4}
\]

where \( H(y) \) and \( G(x) \) are antiderivatives of \( p(y) = 1/h(y) \) and \( g(x) \), respectively.
METHOD OF SOLUTION  Equation (4) indicates the procedure for solving separable equations. A one-parameter family of solutions, usually given implicitly, is obtained by integrating both sides of \( p(y) \, dy = g(x) \, dx \).

NOTE  There is no need to use two constants in the integration of a separable equation, because if we write \( H(y) + c_1 = G(x) + c_2 \), then the difference \( c_2 - c_1 \) can be replaced by a single constant \( c \), as in (4). In many instances throughout the chapters that follow, we will relabel constants in a manner convenient to a given equation. For example, multiples of constants or combinations of constants can sometimes be replaced by a single constant.

EXAMPLE 1  Solving a Separable DE

Solve \((1 + x) \, dy - y \, dx = 0\).

SOLUTION  Dividing by \((1 + x)y\), we can write \( \frac{dy}{y} = \frac{dx}{1 + x} \), from which it follows that

\[
\ln |y| = \ln |1 + x| + c_1
\]

\[
y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \quad \leftarrow \text{laws of exponents}
\]

\[
y = |1 + x| \cdot e^{c_1} = \pm e^{c_1}(1 + x).
\]

Relabeling \( \pm e^{c_1} \) as \( c \) then gives \( y = c(1 + x) \).

ALTERNATIVE SOLUTION  Because each integral results in a logarithm, a judicious choice for the constant of integration is \( \ln|c| \) rather than \( c \). Rewriting the second line of the solution as \( \ln|y| = \ln|1 + x| + \ln|c| \) enables us to combine the terms on the right-hand side by the properties of logarithms. From \( \ln|y| = \ln|c(1 + x)| \) we immediately get \( y = c(1 + x) \). Even if the indefinite integrals are not all logarithms, it may still be advantageous to use \( \ln|c| \). However, no firm rule can be given.

In Section 1.1 we saw that a solution curve may be only a segment or an arc of the graph of an implicit solution \( G(x, y) = 0 \).

EXAMPLE 2  Solution Curve

Solve the initial-value problem \( \frac{dy}{dx} = \frac{x}{y}, \quad y(4) = -3 \).

SOLUTION  Rewriting the equation as \( y \, dy = -x \, dx \), we get

\[
\int y \, dy = -\int x \, dx \quad \text{and} \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.
\]

We can write the result of the integration as \( x^2 + y^2 = c^2 \) by replacing the constant \( 2c_1 \) by \( c^2 \). This solution of the differential equation represents a family of concentric circles centered at the origin.

Now when \( x = 4, \) \( y = -3 \), so \( 16 + 9 = 25 = c^2 \). Thus the initial-value problem determines the circle \( x^2 + y^2 = 25 \) with radius 5. Because of its simplicity we can solve this implicit solution for an explicit solution that satisfies the initial condition.
We saw this solution as \( y = \phi_2(x) \) or \( y = -\sqrt{25 - x^2}, \quad -5 < x < 5 \) in Example 3 of Section 1.1. A solution curve is the graph of a differentiable function. In this case the solution curve is the lower semicircle, shown in dark blue in Figure 2.2.1 containing the point \((4, -3)\).

**LOSING A SOLUTION** Some care should be exercised in separating variables, since the variable divisors could be zero at a point. Specifically, if \( r \) is a zero of the function \( h(y) \), then substituting \( y = r \) into \( \frac{dy}{dx} = g(x)h(y) \) makes both sides zero; in other words, \( y = r \) is a constant solution of the differential equation. But after variables are separated, the left-hand side of \( \frac{dy}{h(y)} = g(x) \, dx \) is undefined at \( r \). As a consequence, \( y = r \) might not show up in the family of solutions that are obtained after integration and simplification. Recall that such a solution is called a singular solution.

**EXAMPLE 3** Losing a Solution

Solve \( \frac{dy}{dx} = y^2 - 4 \).

**SOLUTION** We put the equation in the form

\[
\frac{dy}{y^2 - 4} = dx \quad \text{or} \quad \left[ \frac{1}{y - 2} - \frac{1}{y + 2} \right] dy = dx. \tag{5}
\]

The second equation in (5) is the result of using partial fractions on the left-hand side of the first equation. Integrating and using the laws of logarithms gives

\[
\frac{1}{4} \ln |y - 2| - \frac{1}{4} \ln |y + 2| = x + c_1 \]

or

\[
\ln \left| \frac{y - 2}{y + 2} \right| = 4x + c_2 \quad \text{or} \quad \frac{y - 2}{y + 2} = \pm e^{4x+c_2}.
\]

Here we have replaced \( 4c_1 \) by \( c_2 \). Finally, after replacing \( \pm e^{c_2} \) by \( c \) and solving the last equation for \( y \), we get the one-parameter family of solutions

\[
y = 2 \frac{1 + ce^{4x}}{1 - ce^{4x}}. \tag{6}
\]

Now if we factor the right-hand side of the differential equation as \( dy/dx = (y - 2)(y + 2) \), we know from the discussion of critical points in Section 2.1 that \( y = 2 \) and \( y = -2 \) are two constant (equilibrium) solutions. The solution \( y = 2 \) is a member of the family of solutions defined by (6) corresponding to the value \( c = 0 \). However, \( y = -2 \) is a singular solution; it cannot be obtained from (6) for any choice of the parameter \( c \). This latter solution was lost early on in the solution process. Inspection of (5) clearly indicates that we must preclude \( y = \pm 2 \) in these steps.

**EXAMPLE 4** An Initial-Value Problem

Solve \( (e^y - y) \cos x \frac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0 \).
SOLUTION  Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y} \, dy = \frac{\sin 2x}{\cos x} \, dx.$$  

Before integrating, we use termwise division on the left-hand side and the trigonometric identity $\sin 2x = 2 \sin x \cos x$ on the right-hand side. Then

$$\int (e^y - ye^{-y}) \, dy = 2 \int \sin x \, dx$$

yields

$$e^y + ye^{-y} + e^{-y} = -2 \cos x + c. \quad (7)$$

The initial condition $y = 0$ when $x = 0$ implies $c = 4$. Thus a solution of the initial-value problem is

$$e^y + ye^{-y} + e^{-y} = 4 - 2 \cos x. \quad (8)$$

USE OF COMPUTERS  The Remarks at the end of Section 1.1 mentioned that it may be difficult to use an implicit solution $G(x, y) = 0$ to find an explicit solution $y = \phi(x)$. Equation (8) shows that the task of solving for $y$ in terms of $x$ may present more problems than just the drudgery of symbol pushing—sometimes it simply cannot be done! Implicit solutions such as (8) are somewhat frustrating; neither the graph of the equation nor an interval over which a solution satisfying $y(0) = 0$ is defined is apparent. The problem of “seeing” what an implicit solution looks like can be overcome in some cases by means of technology. One way of proceeding is to use the contour plot application of a computer algebra system (CAS). Recall from multivariate calculus that for a function of two variables $z = G(x, y)$ the two-dimensional curves defined by $G(x, y) = c$, where $c$ is constant, are called the level curves of the function. With the aid of a CAS, some of the level curves of the function $G(x, y) = e^y + ye^{-y} + e^{-y} + 2 \cos x$ have been reproduced in Figure 2.2.2. The family of solutions defined by (7) is the level curves $G(x, y) = c$. Figure 2.2.3 illustrates the level curve $G(x, y) = 4$, which is the particular solution (8), in blue color. The other curve in Figure 2.2.3 is the level curve $G(x, y) = 2$, which is the member of the family $G(x, y) = c$ that satisfies $y(\pi/2) = 0$.

If an initial condition leads to a particular solution by yielding a specific value of the parameter $c$ in a family of solutions for a first-order differential equation, there is a natural inclination for most students (and instructors) to relax and be content. However, a solution of an initial-value problem might not be unique. We saw in Example 4 of Section 1.2 that the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0 \quad (9)$$

has at least two solutions, $y = 0$ and $y = \frac{1}{16} x^4$. We are now in a position to solve the equation. Separating variables and integrating $y^{-1/2} \, dy = x \, dx$ gives

$$2y^{1/2} = \frac{x^2}{2} + c_1 \quad \text{or} \quad y = \left( \frac{x^2}{4} + c \right)^2.$$  

When $x = 0$, then $y = 0$, so necessarily, $c = 0$. Therefore $y = \frac{1}{16} x^4$. The trivial solution $y = 0$ was lost by dividing by $y^{1/2}$. In addition, the initial-value problem (9) possesses infinitely many more solutions, since for any choice of the parameter $a \geq 0$ the

\[In Section 2.6 we will discuss several other ways of proceeding that are based on the concept of a numerical solver.\]
piecewise-defined function
\[ y = \begin{cases} 0, & x < a \\ \frac{1}{12} (x^2 - a^2)^2, & x \geq a \end{cases} \]
satisfies both the differential equation and the initial condition. See Figure 2.2.4.

**SOLUTIONS DEFINED BY INTEGRALS** If \( g \) is a function continuous on an open interval \( I \) containing \( a \), then for every \( x \) in \( I \),
\[ \frac{d}{dx} \int_a^x g(t) \, dt = g(x). \]

You might recall that the foregoing result is one of the two forms of the fundamental theorem of calculus. In other words, \( \int_a^x g(t) \, dt \) is an antiderivative of the function \( g \). There are times when this form is convenient in solving DEs. For example, if \( g \) is continuous on an interval \( I \) containing \( x_0 \) and \( x \), then a solution of the simple initial-value problem \( \frac{dy}{dx} = g(x) \), \( y(x_0) = y_0 \), that is defined on \( I \) is given by
\[ y(x) = y_0 + \int_{x_0}^x g(t) \, dt \]

You should verify that \( y(x) \) defined in this manner satisfies the initial condition. Since an antiderivative of a continuous function \( g \) cannot always be expressed in terms of elementary functions, this might be the best we can do in obtaining an explicit solution of an IVP. The next example illustrates this idea.

**EXAMPLE 5** An Initial-Value Problem

Solve \( \frac{dy}{dx} = e^{-x} \), \( y(3) = 5 \).

**SOLUTION** The function \( g(x) = e^{-x} \) is continuous on \((-\infty, \infty)\), but its antiderivative is not an elementary function. Using \( t \) as dummy variable of integration, we can write
\[ \int_3^x \frac{dy}{dt} \, dt = \int_3^x e^{-t} \, dt \]
\[ y(t) \bigg|_3^x = \int_3^x e^{-t} \, dt \]
\[ y(x) - y(3) = \int_3^x e^{-t} \, dt \]
\[ y(x) = y(3) + \int_3^x e^{-t} \, dt. \]

Using the initial condition \( y(3) = 5 \), we obtain the solution
\[ y(x) = 5 + \int_3^x e^{-t} \, dt. \]

The procedure demonstrated in Example 5 works equally well on separable equations \( \frac{dy}{dx} = g(x) f(y) \) where, say, \( f(y) \) possesses an elementary antiderivative but \( g(x) \) does not possess an elementary antiderivative. See Problems 29 and 30 in Exercises 2.2.
In Problems 1–22 solve the given differential equation by separation of variables.

1. \( \frac{dy}{dx} = \sin 5x \)
2. \( \frac{dy}{dx} = (x + 1)^2 \)
3. \( dx + e^{3x} dy = 0 \)
4. \( dy - (y - 1)^2 dx = 0 \)
5. \( x \frac{dy}{dx} = 4y \)
6. \( \frac{dy}{dx} + 2xy^2 = 0 \)
7. \( \frac{dy}{dx} = e^{3x} + 2y \)
8. \( e^y \frac{dy}{dx} = e^{-y} + e^{-2x-y} \)
9. \( \ln x \frac{dx}{dy} = \left( \frac{y + 1}{x} \right)^2 \)
10. \( \frac{dy}{dx} = \frac{(2y + 3)^2}{(4x + 5)} \)
11. \( \csc y \ dx + \sec^2 x \ dy = 0 \)
12. \( \sin 3x \ dx + 2y \cos^3 3x \ dy = 0 \)
13. \( (e^y + 1)^2 e^{-y} \ dx + (e^y + 1)^3 e^{-x} \ dy = 0 \)
14. \( x(1 + y^2)^{1/2} \ dx = y(1 + x^2)^{1/2} \ dy \)
15. \( \frac{dS}{dr} = kS \)
16. \( \frac{dQ}{dt} = k(Q - 70) \)
17. \( \frac{dP}{dt} = P - P^2 \)
18. \( \frac{dN}{dt} + N = Nte^{t+2} \)
19. \( \frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \)
20. \( \frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3y + x - 3} \)
21. \( \frac{dy}{dx} = x\sqrt{1 - y^2} \)
22. \( (e^y + e^{-x}) \frac{dy}{dx} = y^2 \)

In Problems 23–28 find an explicit solution of the given initial-value problem.

23. \( \frac{dx}{dt} = 4(x^2 + 1), \ x(\pi/4) = 1 \)
24. \( \frac{dy}{dx} = \frac{y^2 - 1}{x^2 - 1}, \ y(2) = 2 \)
25. \( \frac{dy}{dx} = y - xy, \ y(-1) = -1 \)
26. \( \frac{dy}{dt} + 2y = 1, \ y(0) = \frac{5}{2} \)
27. \( \sqrt{1 - y^2} \ dx - \sqrt{1 - x^2} \ dy = 0, \ y(0) = \frac{\sqrt{3}}{2} \)
28. \( (1 + x^4) \ dy + x(1 + 4y^2) \ dx = 0, \ y(1) = 0 \)

In Problems 29 and 30 proceed as in Example 5 and find an explicit solution of the given initial-value problem.

29. \( \frac{dy}{dx} = ye^{-x^2}, \ y(4) = 1 \)
30. \( \frac{dy}{dt} = y^2 \sin x^2, \ y(-2) = \frac{1}{3} \)
31. (a) Find a solution of the initial-value problem consisting of the differential equation in Example 3 and the initial conditions \( y(0) = 2, \ y(0) = -2, \) and \( y(0) = 1. \)
\( (b) \) Find the solution of the differential equation in Example 4 when \( \ln c_1 \) is used as the constant of integration on the left-hand side in the solution and \( 4 \ln c_1 \) is replaced by \( \ln c \). Then solve the same initial-value problems in part (a).

32. Find a solution of \( x \frac{dy}{dx} = y^2 - y \) that passes through the indicated points.
   \[(a) \quad (0, 1) \quad (b) \quad (0, 0) \quad (c) \quad \left( \frac{1}{2}, \frac{1}{2} \right) \quad (d) \quad \left( 2, \frac{1}{2} \right) \]


34. Show that an implicit solution of
   \[ 2x \sin^2 y \, dx - (x^2 + 10) \cos y \, dy = 0 \]
   is given by \( \ln(x^2 + 10) + \csc y = c \). Find the constant solutions, if any, that were lost in the solution of the differential equation.

Often a radical change in the form of the solution of a differential equation corresponds to a very small change in either the initial condition or the equation itself. In Problems 35–38 find an explicit solution of the given initial-value problem. Use a graphing utility to plot the graph of each solution. Compare each solution curve in a neighborhood of \((0, 1)\).

35. \( \frac{dy}{dx} = (y - 1)^2, \quad y(0) = 1 \)

36. \( \frac{dy}{dx} = (y - 1)^2, \quad y(0) = 1.01 \)

37. \( \frac{dy}{dx} = (y - 1)^2 + 0.01, \quad y(0) = 1 \)

38. \( \frac{dy}{dx} = (y - 1)^2 - 0.01, \quad y(0) = 1 \)

39. Every autonomous first-order equation \( \frac{dy}{dx} = f(y) \) is separable. Find explicit solutions \( y_1(x), y_2(x), y_3(x), \) and \( y_4(x) \) of the differential equation \( \frac{dy}{dx} = y - y^3 \) that satisfy, in turn, the initial conditions \( y_1(0) = 2, \quad y_2(0) = \frac{1}{2}, \quad y_3(0) = -\frac{1}{2}, \) and \( y_4(0) = -2 \). Use a graphing utility to plot the graphs of each solution. Compare these graphs with those predicted in Problem 19 of Exercises 2.1. Give the exact interval of definition for each solution.

40. (a) The autonomous first-order differential equation \( \frac{dy}{dx} = 1/(y - 3) \) has no critical points. Nevertheless, place 3 on the phase line and obtain a phase portrait of the equation. Compute \( d^2y/dx^2 \) to determine where solution curves are concave up and where they are concave down (see Problems 35 and 36 in Exercises 2.1). Use the phase portrait and concavity to sketch, by hand, some typical solution curves.

(b) Find explicit solutions \( y_1(x), y_2(x), y_3(x), \) and \( y_4(x) \) of the differential equation in part (a) that satisfy, in turn, the initial conditions \( y_1(0) = 4, \quad y_2(0) = 2, \quad y_3(1) = 2, \) and \( y_4(-1) = 4 \). Graph each solution and compare with your sketches in part (a). Give the exact interval of definition for each solution.

41. (a) Find an explicit solution of the initial-value problem
   \[ \frac{dy}{dx} = \frac{2x + 1}{2y}, \quad y(-2) = -1. \]

(b) Use a graphing utility to plot the graph of the solution in part (a). Use the graph to estimate the interval \( I \) of definition of the solution.

(c) Determine the exact interval \( I \) of definition by analytical methods.

42. Repeat parts (a)–(c) of Problem 41 for the IVP consisting of the differential equation in Problem 7 and the initial condition \( y(0) = 0 \).

Discussion Problems

43. (a) Explain why the interval of definition of the explicit solution \( y = \phi_2(x) \) of the initial-value problem in Example 2 is the open interval \((-5, 5)\).

(b) Can any solution of the differential equation cross the \( x \)-axis? Do you think that \( x^2 + y^2 = 1 \) is an implicit solution of the initial-value problem \( \frac{dy}{dx} = -x/y, \quad y(1) = 0 \)?

44. (a) If \( a > 0 \), discuss the differences, if any, between the solutions of the initial-value problems consisting of the differential equation \( \frac{dy}{dx} = x/y \) and each of the initial conditions \( y(a) = a, \quad y(-a) = -a, \) and \( y(-a) = -a \).

(b) Does the initial-value problem \( \frac{dy}{dx} = x/y, \quad y(0) = 0 \) have a solution?

(c) Solve \( \frac{dy}{dx} = x/y, \quad y(1) = 2 \) and give the exact interval \( I \) of definition of its solution.

45. In Problems 39 and 40 we saw that every autonomous first-order differential equation \( \frac{dy}{dx} = f(y) \) is separable. Does this fact help in the solution of the initial-value problem \( \frac{dy}{dx} = \sqrt{1 + y^2} \sin^2 y, \quad y(0) = \frac{1}{2} \)? Discuss. Sketch, by hand, a plausible solution curve of the problem.

46. Without the use of technology, how would you solve
   \[ (\sqrt{x} + x) \frac{dy}{dx} = \sqrt{y} + y \]

Carry out your ideas.

47. Find a function whose square plus the square of its derivative is 1.

48. (a) The differential equation in Problem 27 is equivalent to the normal form
   \[ \frac{dy}{dx} = \sqrt{1 - y^2} \]
in the square region in the \(xy\)-plane defined by \(|x| < 1, |y| < 1\). But the quantity under the radical is nonnegative also in the regions defined by \(|x| > 1, |y| > 1\). Sketch all regions in the \(xy\)-plane for which this differential equation possesses real solutions.

(b) Solve the DE in part (a) in the regions defined by \(|x| > 1, |y| > 1\). Then find an implicit and an explicit solution of the differential equation subject to \(y(2) = 2\).

### Mathematical Model

**49. Suspension Bridge** In (16) of Section 1.3 we saw that a mathematical model for the shape of a flexible cable strung between two vertical supports is

\[
\frac{dy}{dx} = \frac{W}{T_1}
\]

where \(W\) denotes the portion of the total vertical load between the points \(P_1\) and \(P_2\) shown in Figure 1.3.7. The DE (10) is separable under the following conditions that describe a suspension bridge.

Let us assume that the \(x\)- and \(y\)-axes are as shown in Figure 2.2.5—that is, the \(x\)-axis runs along the horizontal roadbed, and the \(y\)-axis passes through \((0, a)\), which is the lowest point on one cable over the span of the bridge, coinciding with the interval \([-\frac{L}{2}, \frac{L}{2}]\). In the case of a suspension bridge, the usual assumption is that the vertical load in (10) is only a uniform roadbed distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed (say, pounds per horizontal foot) is a constant \(\rho\). Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation \(y = \phi(x)\)) of each of the two cables in a suspension bridge is determined. Express your solution of the IVP in terms of the sag \(h\) and span \(L\). See Figure 2.2.5.

**FIGURE 2.2.5** Shape of a cable in Problem 49

### Computer Lab Assignments

**50. (a)** Use a CAS and the concept of level curves to plot representative graphs of members of the family of solutions of the differential equation

\[
\frac{dy}{dx} = -\frac{8x + 5}{3y^2 + 1}
\]

Experiment with different numbers of level curves as well as various rectangular regions defined by \(a \leq x \leq b, c \leq y \leq d\).

**51. (a)** Find an implicit solution of the IVP

\[
(2y + 2)dy - (4x^3 + 6x)dx = 0, \quad y(0) = -3.
\]

(b) Use part (a) to find an explicit solution \(y = \phi(x)\) of the IVP.

(c) Consider your answer to part (b) as a function only. Use a graphing utility or a CAS to graph this function, and then use the graph to estimate its domain.

(d) With the aid of a root-finding application of a CAS, determine the approximate largest interval \(I\) of definition of the solution \(y = \phi(x)\) in part (b). Use a graphing utility or a CAS to graph the solution curve for the IVP on this interval.

**52. (a)** Use a CAS and the concept of level curves to plot representative graphs of members of the family of solutions of the differential equation

\[
\frac{dy}{dx} = \frac{x(1-x)}{y(-2+y)}
\]

Experiment with different numbers of level curves as well as various rectangular regions in the \(xy\)-plane until your result resembles Figure 2.2.6.

(b) On separate coordinate axes, plot the graph of the implicit solution corresponding to the initial condition \(y(0) = \frac{1}{2}\). Use a colored pencil to mark off that segment of the graph that corresponds to the solution curve of a solution \(\phi\) that satisfies the initial condition. With the aid of a root-finding application of a CAS, determine the approximate largest interval \(I\) of definition of the solution \(\phi\). [Hint: First find the points on the curve in part (a) where the tangent is vertical.]

(c) Repeat part (b) for the initial condition \(y(0) = -2\).

**FIGURE 2.2.6** Level curves in Problem 52