Appendix Supplement for the paper "Semiparametric Estimation of a Simultaneous Game with Incomplete Information"

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Abstract

We present a direct, step by step proof of Theorem A-1 in the paper *Semiparametric Estimation of a Simultaneous Game with Incomplete Information*

Suppose $(X, Z) \in \mathbb{R}^P \times \mathbb{R}^L$ is a random vector with joint density $f_{X,Z}(x, z)$ and let $M \geq L+1$. Assume an iid sample $\{X_n, Z_n\}_{n=1}^N$. Fix $\gamma \in \mathbb{R}^D$ and $z \in \mathbb{R}^L$, consider a function $\eta : \mathbb{R}^P \times \mathbb{R}^L \times \mathbb{R}^D \to \mathbb{R}$, a kernel $K : \mathbb{R}^L \to \mathbb{R}$ and a bandwidth $h_N \to 0$. Let $K_{h_N}(\psi) = K\big(\psi/h_N\big)$ and define $R_N(z,\gamma) = \big(Nh_N^L\big)^{-1}\sum_{n=1}^N \eta(X_n, z, \gamma) K_{h_N}(Z_n - z)$, $\widehat{f}_{Z_N}(z) = \big(Nh_N^L\big)^{-1}\sum_{n=1}^N K_{h_N}(Z_n - z)$ and $\mu_N(z,\gamma) = R_N(z,\gamma)/\widehat{f}_{Z_N}(z)$. For any $z \in \mathbb{S}(Z)$ let $\mu(z,\gamma) = E\big[\eta(X, z, \gamma)\big|\, Z = z\big]$. Consider the following assumptions:

Assumption S1. (A) Z is absolutely continuous w.r.t Lebesgue measure. (B) $f_{X,Z}(x, z)$ and $f_Z(z)$ are bounded, M times differentiable with respect to z with bounded derivatives.

Assumption S2. There exist compact sets $\mathcal{Z} \subset \mathbb{S}(Z)$ with $\inf_{z\in\mathcal{Z}} f_Z(z) > 0$, and $\Gamma \subset \mathbb{R}^D$ such that: (A) $\mu(z,\gamma)$ is M times differentiable w.r.t z and γ with bounded derivatives $\forall\ z \in \mathbb{S}(Z)$, $\gamma \in \Gamma$. (B) There exists $\overline{\eta} : \mathbb{R}^P \to \mathbb{R}_+$ such that $\big|\eta(X, z, \gamma)\big| \leq \overline{\eta}(X)$ w.p.1 for all $X \in \mathbb{S}(X)$, $z \in \mathcal{Z}$, $\gamma \in \Gamma$; $E\big[\overline{\eta}(X)^2\big|\, Z = z\big]$ is a continuous function of z for all $z \in \mathbb{S}(Z)$, and $E\big[\overline{\eta}(X)^4\big] < \infty$. (C) There exists $\overline{\eta}_1 : \mathbb{R}^P \to \mathbb{R}_+$, and $\varphi_1 > 0$ such that $\big|\eta(X, z, \gamma) - \eta(X, z', \gamma)\big| \leq \overline{\eta}_1(X)\big\|z - z'\big\|^{\varphi_1}$ w.p.1 for all $X \in \mathbb{S}(X)$, $z, z' \in \mathcal{Z}$, $\gamma \in \Gamma$, and $E\big[\overline{\eta}_1(X)\big] < \infty$. (D) There exists $\overline{\eta}_2 : \mathbb{R}^P \to \mathbb{R}_+$, and $\varphi_2 > 0$ such that $\big|\eta(X, z, \gamma) - \eta(X, z, \gamma')\big| \leq \overline{\eta}_2(X)\big\|\gamma - \gamma'\big\|^{\varphi_2}$ w.p.1 for all $X \in \mathbb{S}(X)$, $z \in \mathcal{Z}$, $\gamma, \gamma' \in \Gamma$, and $E\big[\overline{\eta}_2(X)\big] < \infty$.

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Assumption S3. (A) The kernel $K(\cdot)$ has compact support, is Lipschitz-continuous, bounded and symmetric about zero. Denote $\psi = (\psi_1, \ldots, \psi_L)'$, then $\int K(\psi)d\psi = 1$, $\int \|\psi\|^M |K(\psi)| d\psi < \infty$ and $\int (\psi_1^{q_1} \cdots \psi_L^{q_L}) K(\psi) d\psi_1 \ldots d\psi_L = 0$ for all $0 < q_1 + \cdots + q_L < M$. (B) $h_{_N} \to 0$ satisfies: $Nh_{_N}^{L+2} \to \infty$; $Nh_{_N}^{2L}/\log(N) \to \infty$ and $Nh_{_N}^{2M} \to 0$. [1]

Theorem A-1 *If assumptions S1-S3 are satisfied, then for any $z \in \mathcal{Z}$, $\gamma \in \Gamma$,*

$$\mu_{_N}(z,\gamma) - \mu(z,\gamma) = \frac{1}{f_{_Z}(z)} \frac{1}{Nh_{_N}^L} \sum_{n=1}^{N} \big[\eta(X_n, z, \gamma) - \mu(z,\gamma)\big] K_{h_{_N}}(Z_n - z) + \xi_N(z,\gamma)$$

where $\sup_{\substack{z \in \mathcal{Z} \\ \gamma \in \Gamma}} \big|\xi_N(z,\gamma)\big| = O_p\big(N^{\delta-1} h_{_N}^{-L}\big)$ *for any $\delta > 0$.*

Corollary 1 If we strengthen the condition $\log N h_{_N}^{-2L} = o(N)$ to $N^\delta h_{_N}^{-2L} = o(N)$ for some $\delta > 0$. Let $\xi_N(z,\gamma)$ be as defined in Theorem A-1, then $\sup_{\substack{z \in \mathcal{Z} \\ \gamma \in \Gamma}} \big|\xi_N(z,\gamma)\big| = o_p(\mathrm{N}^{-1/2})$.

Proof of Theorem A-1: Let $\varphi = \text{Min}\,\{1, \varphi_{_1}, \varphi_{_2}\}$. Without loss of generality, suppose $\mathcal{Z} = [a_1, b_1] \times \cdots \times [a_L, b_L]$ and $\Gamma = [e_1, h_1] \times \cdots \times [e_D, h_D]$ where $a_\ell < b_\ell$ and $e_d < h_d$.[2] For $\ell = 1, \ldots, L$ and $d = 1, \ldots, D$, let $z_0^{(\ell)} = a_\ell$, $\gamma_0^{(d)} = e_d$, $z_i^{(\ell)} = \text{Min}\{z_0^{(\ell)} + i/N^{1/\varphi}\ , b_\ell\}$ and $\gamma_j^{(d)} = \text{Min}\{\gamma_0^{(d)} + j/N^{1/\varphi}\ , h_\ell\}$ where $i, j \in \mathbb{N}$. Define the sets $\mathcal{A}_{1_N} \subset \mathcal{Z}$ and $\mathcal{A}_{2_N} \subset \Gamma$ as $\mathcal{A}_{1_N} = \{z_0^{(1)}, \ldots, z_{Q_1}^{(1)}\} \times \cdots \times \{z_0^{(L)}, \ldots, z_{Q_L}^{(L)}\}$ and $\mathcal{A}_{2_N} = \{\gamma_0^{(1)}, \ldots, \gamma_{T_1}^{(1)}\} \times \cdots \times \{\gamma_0^{(D)}, \ldots, \gamma_{T_D}^{(D)}\}$. Let $z^* = \max_{z \in \mathcal{Z}} \|z\|$ and $\gamma^* = \max_{\gamma \in \Gamma} \|\gamma\|$. It follows that $Q_{_\ell} \leq \lceil 2z^* N^{1/\varphi} \rceil\ \forall\ \ell$, $T_{_d} \leq \lceil 2\gamma^* N^{1/\varphi} \rceil\ \forall\ d$; $\#\mathcal{A}_{1_N} < (2(z^*+1))^L N^{L/\varphi}$ and $\#\mathcal{A}_{2_N} < (2(\gamma^*+1))^D N^{D/\varphi}$ for all N. For any $(z,\gamma) \in \mathcal{Z} \times \Gamma$ we will denote from now on: $z_\kappa = \underset{u \in \mathcal{A}_{1_N}}{\text{argmin}}\ \|u - z\|$ and $\gamma_\kappa = \underset{v \in \mathcal{A}_{2_N}}{\text{argmin}}\ \|v - \gamma\|$. Note that $\sup_{z \in \mathcal{Z}} \|z - z_\kappa\| \leq \sqrt{L}/N^{1/\varphi}$ and $\sup_{\gamma \in \Gamma} \|\gamma - \gamma_\kappa\| \leq \sqrt{D}/N^{1/\varphi}$ by construction.

Step 1 Take any pair of random variables $\mathcal{S}_{_N}, \mathcal{R}_{_N}$ such that: $\mathcal{S}_{_N} \leq \mathcal{R}_{_N}$ and $\mathcal{S}_{_N} \in [0,1]$ w.p.1 $\forall\ N$. Suppose there exist $\varepsilon_1 \in (0,1)$, $\varepsilon_2 \in (0,1)$ and $\overline{N}$ such that $Pr(\mathcal{R}_{_N} > \varepsilon_1) \leq \varepsilon_2\ \forall\ N \geq \overline{N}$. Then, $E[\mathcal{S}_N] \leq \varepsilon_1 + \varepsilon_2\ \ \forall\ N \geq \overline{N}$.

Proof: $E[S_N] \leq \varepsilon_1 \cdot \Pr(S_N \leq \varepsilon_1) + 1 \cdot \Pr(S_N > \varepsilon_1) \leq \varepsilon_1 \cdot 1 + 1 \cdot \Pr(R_{_N} > \varepsilon_1) \leq \varepsilon_1 + \varepsilon_2\ \forall\ N \geq \overline{N}$.

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[1] If $L \geq 2$, $Nh_{_N}^{2L}/\log(N) \to \infty$ implies $Nh_{_N}^{L+2} \to \infty$.

[2] Every pair compact sets in $\mathbb{R}^L$ and $\mathbb{R}^D$ with Lebesgue measure greater than zero contains a set of the form $[a_1, b_1] \times \cdots \times [a_L, b_L]$ and $[e_1, h_1] \times \cdots \times [e_D, h_D]$ respectively, where $a_\ell < b_\ell$ and $e_d < h_d$.

Step 2 Define the objects

$$V_{1_N}(z) = \left(Nh_N^L\right)^{-1}\sum_{n=1}^{N}\overline{\eta}(X_n)^2 K_{h_N}(Z_n - z)^2 \quad \text{and} \quad V_{2N}(z) = N^{-1}\sum_{n=1}^{N}\overline{\eta}(X_n)\big|K_{h_N}(Z_n - z)\big|.$$

Then $\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ V_{1_N}(z) = O_p(1)$ and $\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ V_{2N}(z) = O_p(1)$.

Proof: By continuity of $E\big[\overline{\eta}(X)\big|Z\big]$ and boundedness of $K(\cdot)$, $\exists\ \overline{K}$ and $\overline{V}_1$ such that $\max_{\psi\in\mathbb{R}^L}\big|K(\psi)\big| < \overline{K}$ and $\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ EV_{1_N}(z)$. Define $W_{1_N} = \overline{K}^2\overline{\eta}(X_n)^2 + h_N^L\overline{V}_1$ and $\overline{W_{1_N}^2} = N^{-1}\sum_{n=1}^N W_{1_N}^2$. Existence of $E\big[\overline{\eta}(X)^4\big]$ implies that $\overline{W_{1_N}^2} = O_p(1)$. Take any $\overline{M} > 0$. Using Hoeffding's inequality and the fact that $\#\mathcal{A}_{1_N} < (2(z^*+1))^L N^{L/\varphi}$, S1-S3 yield $\Pr\left(\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ \big|V_{1_N}(z) - EV_{1_N}(z)\big| > M\right) \leq \sum_{z\in\mathcal{A}_{1_N}}\Pr\left(\big|V_{1_N}(z) - EV_{1_N}(z)\big| > M\right) < 2(2(z^*+1))^L N^{L/\varphi}\exp\left\{-\frac{1}{2}Nh_N^{2L}M^2\Big/\overline{W_{1_N}^2}\right\}$. Let $a_{1_N} = \log(2) + L\cdot\log(2(z^*+1)) + (L/\varphi)\log(N)$. Take any $\varepsilon\in(0,1)$. Since $\overline{W_{1_N}^2} = O_p(1)$, there exists $\overline{N}_\varepsilon$ and $\Delta_\varepsilon > 0$ such that $\Pr\left(\overline{W_{1_N}^2} > \Delta_\varepsilon\right) < \varepsilon/2$ for all $N > \overline{N}_\varepsilon$. Define $M_\varepsilon = \sqrt{2\Delta_\varepsilon\big(a_{1_{\overline{N}_\varepsilon}} - \log(\varepsilon/2)\big)\big/\overline{N}_\varepsilon h_{\overline{N}_\varepsilon}^{2L}}$. Since $Nh_N^{2L}/\log(N) \to \infty$, we have $a_{1_N} - \frac{1}{2}Nh_N^{2L}M_\varepsilon^2\Big/\Delta_\varepsilon < \log(\varepsilon/2)\ \ \forall\ N > \overline{N}_\varepsilon$. Therefore $\forall\ \varepsilon\in(0,1)$, $\exists\ M_\varepsilon, \overline{N}_\varepsilon$ such that $\Pr\left(2(2(z^*+1))^L N^{L/\varphi}\exp\left\{-\frac{1}{2}Nh_N^{2L}M_\varepsilon^2\Big/\overline{W_{1_N}^2}\right\} > \varepsilon/2\right) < \varepsilon/2$. Then $\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ V_{1_N}(z) = O_p(1)$ follows from Step 1 with $\mathcal{S}_N = \Pr\left(\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ \big|V_{1_N}(z) - EV_{1_N}(z)\big| > M_\varepsilon\right)$ and $\mathcal{R}_N = 2(2(z^*+1))^L N^{L/\varphi}\exp\left\{-\frac{1}{2}Nh_N^{2L}M_\varepsilon^2\Big/\overline{W_{1_N}^2}\right\}$. The result $\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ V_{2N}(z) = O_p(1)$ follows more simply by noting that $\underset{z\in\mathcal{A}_{1_N}}{\text{Max}}\ V_{2_N}(z) \leq \overline{K}N^{-1}\sum_{n=1}^N\overline{\eta}(X_n) = O_p(1)$. □

Step 3 If Assumptions S1-S3 are satisfied, then there exists N' and $\overline{R}$ such that for all $N > N'$: $\underset{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}}{sup}\left|ER_N(z,\gamma) - f_Z(z)\mu(z,\gamma)\right| \leq h_N^M\overline{R}$.

Proof: Take any $(z,\gamma)\in\mathcal{Z}\times\Gamma$. Given our assumptions, $\exists\ C > 0$ and $N'\in\mathbb{N}$ such that $\forall\ N > N'$, an M^{th}-order Taylor approximation yields

$$\sup_{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}}\left|ER_N(z,\gamma) - f_Z(z)\mu(z,\gamma)\right| \leq C\frac{h_N^M}{M!}\left|\int\sum_{Q_M}\psi_1^{q_1}\cdots\psi_L^{q_L}K(\psi)d\psi\right|.$$

The result follows from the fact that $\int\big\|\psi\big\|^M\big|K(\psi)\big|d\psi < \infty$. □

Step 4 $\underset{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}}{sup}\ \left(N^{1-\delta}h_N^L\right)^{1/2}\left|R_N(z,\gamma) - ER_N(z,\gamma)\right| = O_p(1)$ for any $\delta > 0$.

Proof: Let z_κ and γ_κ be as defined prior to Step 1. The triangle inequality yields

$$\begin{aligned}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| \ \le\ & \Big|R_N(z_\kappa,\gamma_\kappa) - ER_N(z_\kappa,\gamma_\kappa)\Big| + \Big|R_N(z,\gamma) - R_N(z_\kappa,\gamma_\kappa)\Big| \\ & + \Big|ER_N(z,\gamma) - ER_N(z_\kappa,\gamma_\kappa)\Big|.\end{aligned} \tag{A-1}$$

By S1-S3: $\sup_{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}} \big(N^{1-\delta}h_N^L\big)^{1/2} \Big|R_N(z,\gamma) - R_N(z_\kappa,\gamma_\kappa)\Big| \le c_k\big(N^{1+\delta}h_N^{L+2}\big)^{-1/2}\sum_{n=1}^N \overline{\eta}(X_n)/N + \overline{K}/\big(N^{1+\delta}h_N^L\big)^{-1/2}\Big[L^{\varphi_1/2}\cdot\sum_{n=1}^N \overline{\eta}_1(X_n)/N + L^{\varphi_2/2}\cdot\sum_{n=1}^N \overline{\eta}_1(X_n)/N\Big] = o_p(1)$. Step 3 yields $\sup_{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}} \big(N^{1-\delta}h_N^L\big)^{1/2}\Big|ER_N(z,\gamma) - ER_N(z_\kappa,\gamma_\kappa)\Big| \le 2\big(N^{1-\delta}h_N^{L+2M}\big)^{1/2}\overline{R} + \big(h_N^L/N^{1+\delta}\big)^{1/2}\cdot\big[\overline{f}c_1 + c_2\big] = o(1)$.

Equation A-1 becomes

$$\sup_{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}} \big(N^{1-\delta}h_N^L\big)^{1/2}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| \le \max_{\substack{z\in\mathcal{A}_{1_N}\\ \gamma\in\mathcal{A}_{2_N}}} \big(N^{1-\delta}h_N^L\big)^{1/2}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| + o_p(1).$$

Take any $M > 0$, then

$$\begin{aligned}&\Pr\Big(\max_{\mathcal{A}_{1_N},\mathcal{A}_{2_N}} \big(N^{1-\delta}h_N^L\big)^{1/2}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| > M\Big) \\ &\qquad\le \sum_{\gamma\in\mathcal{A}_{2_N}}\sum_{z\in\mathcal{A}_{1_N}} \Pr\Big(\big(N^{1-\delta}h_N^L\big)^{1/2}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| > M\Big).\end{aligned}$$

Let $V_N(z) = V_{1_N}(z) + 2\big(h_N^M\overline{R} + \overline{f}\overline{\mu}\big)V_{2_N}(z) + h_N^L\big(h_N^M\overline{R} + \overline{f}\overline{\mu}\big)^2$ and $V_N = \max_{z\in\mathcal{A}_{1_N}} V_N(z)$, where $V_{1_N}(z)$ and $V_{2_N}(z)$ are as in Step 2, $\overline{\mu} = \sup_{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}} \big|\mu(z,\gamma)\big|$ and $\overline{f}$, $\overline{R}$ are as defined above. Using Steps 1, 2 and Hoeffding's inequality, $\Pr\Big(\big(N^{1-\delta}h_N^L\big)^{1/2}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| > M\Big) \le \exp\Big\{-\frac{1}{2}NM^2\big(N^{1-\delta}h_N^L\big)^{-1}\Big/\frac{V_N(z)}{h_N^L}\Big\} = \exp\Big\{-\frac{1}{2}N^\delta M^2\Big/V_N(z)\Big\}\ \forall\ z \in \mathcal{Z}, \gamma \in \Gamma \le \exp\Big\{-\frac{1}{2}N^\delta M^2\Big/V_N\Big\}\ \forall\ z \in \mathcal{A}_{1_N}, \gamma \in \Gamma$. Since $\mathcal{A}_{2_N} \subset \Gamma$, this implies that

$$\begin{aligned}\Pr\Big(\max_{\substack{z\in\mathcal{A}_{1_N}\\ \gamma\in\mathcal{A}_{2_N}}} \big(N^{1-\delta}h_N^L\big)^{1/2}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| > M\Big) &\le \sum_{\gamma\in\mathcal{A}_{2_N}}\sum_{z\in\mathcal{A}_{1_N}} \exp\Big\{-\frac{1}{2}N^\delta M^2\Big/V_N\Big\} \\ &< (2(z^*+1))^L(2(\gamma^*+1))^D N^{(L+D)/\varphi}\exp\Big\{-\frac{1}{2}N^\delta M^2\Big/V_N\Big\},\end{aligned} \tag{A-2}$$

where z^* and γ^* were defined above. From Step 2, we have $V_N = O_p(1)$. Complete the proof by invoking the result of Step 1 and the same arguments as in Step 2, defining a_N and M_ε in the same fashion and letting $\mathcal{S}_N = \Pr\Big(\max_{\substack{z\in\mathcal{A}_{1_N}\\ \gamma\in\mathcal{A}_{2_N}}} \big(N^{1-\delta}h_N^L\big)^{1/2}\Big|R_N(z,\gamma) - ER_N(z,\gamma)\Big| > M_\varepsilon\Big)$ and $\mathcal{R}_N = (2(z^*+1))^L(2(\gamma^*+1))^D N^{(L+D)/\varphi}\exp\Big\{-\frac{1}{2}N^\delta M^2\Big/V_N\Big\}$. □

Step 5 $\sup_{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}} \left(N^{1-\delta}h_N^L\right)^{1/2}\Big|R_N(z,\gamma) - f_Z(z)\mu(z,\gamma)\Big| = O_p(1)$ for any $\delta > 0$.

Proof: Follows immediately from Steps 3, 4 and the bandwidth condition $Nh_N^{2M} \to 0$. □

Step 6 (final step) Using Step 4, $\sup_{z\in\mathcal{Z}}\left(N^{1-\delta}h_N^L\right)^{1/2}\big|\widehat{f}_{Z_N}(z) - f_Z(z)\big| = O_p(1)$ for any $\delta > 0$. Take any $z \in \mathcal{Z}$, $\gamma \in \Gamma$. Consider the second-order approximation

$$\mu_N(z,\gamma) - \mu(z,\gamma) = \frac{1}{f_Z(z)}\big[R_N(z,\gamma) - f_Z(z)\mu(z,\gamma)\big] - \frac{\mu(z,\gamma)}{f_Z(z)}\big[\widehat{f}_{Z_N}(z) - f_Z(z)\big]$$
$$+\frac{1}{2}\big[R_N(z,\gamma) - f_Z(z)\mu(z,\gamma)\ ,\ \widehat{f}_{Z_N}(z) - f_Z(z)\big]\underbrace{\begin{bmatrix} 0 & \frac{-1}{\widetilde{f}_{Z_N}(z)^2} \\ \frac{-1}{\widetilde{f}_{Z_N}(z)^2} & \frac{2\widetilde{R}_N(z,\gamma)}{\widetilde{f}_{Z_N}(z)^3}\end{bmatrix}}_{\equiv \widetilde{H}_N(z,\gamma)}\begin{bmatrix} R_N(z,\gamma) - f_Z(z)\mu(z,\gamma) \\ \widehat{f}_{Z_N}(z) - f_Z(z)\end{bmatrix},$$

with $\widetilde{f}_{Z_N}(z)$ between $f_N(z)$ and $f_Z(z)$, and $\widetilde{R}_N(z,\gamma)$ between $R_N(z,\gamma)$ and $f_Z(z)\mu(z,\gamma)$. Using Step 5 and the characteristics of $\mathcal{Z}$ we get $\sup_{\substack{z\in\mathcal{Z}\\ \gamma\in\Gamma}}\big\|\widetilde{H}_N(z,\gamma)\big\| = O_p(1)$. Given this, the result of Theorem A-1 follows immediately from Step 5. □