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Nonparametric probability bounds for Nash equilibrium actions in a simultaneous discrete game

Andres Aradillas-Lopez
Department of Economics, University of Wisconsin–Madison

We study a simultaneous, complete-information game played by $p = 1, \dots, P$ agents. Each p has an *ordinal* decision variable $Y_p \in \mathcal{A}_p = \{0, 1, \dots, M_p\}$, where M_p can be unbounded, $\mathcal{A}_p$ is p's *action space*, and each element in $\mathcal{A}_p$ is an *action*, that is, a potential value for Y_p. The collective action space is the Cartesian product $\mathcal{A} = \prod_{p=1}^{P} \mathcal{A}_p$. A profile of actions $\mathbf{y} \in \mathcal{A}$ is a Nash equilibrium (NE) profile if $\mathbf{y}$ is played with positive probability in *some* existing NE. Assuming that we observe NE behavior in the data, we characterize bounds for the probability that a prespecified $\mathbf{y}$ in $\mathcal{A}$ is a NE profile. Comparing the resulting upper bound with $\Pr[\mathbf{Y} = \mathbf{y}]$ (where $\mathbf{Y}$ is the observed outcome of the game), we also obtain a lower bound for the probability that the underlying equilibrium selection mechanism $\mathcal{M}_{\mathscr{E}}$ chooses a NE where $\mathbf{y}$ is played *given that such a NE exists*. Our bounds are nonparametric, and they rely on shape restrictions on payoff functions and on the assumption that the researcher has ex ante knowledge about the *direction* of strategic interaction (e.g., that for $q \neq p$, higher values of Y_q reduce p's payoffs). Our results allow us to investigate whether certain action profiles in $\mathcal{A}$ are scarcely observed as outcomes in the data because they are rarely NE profiles *or* because $\mathcal{M}_{\mathscr{E}}$ rarely selects such NE. Our empirical illustration is a multiple entry game played by Home Depot and Lowe's.

Keywords. Ordered response game, nonparametric identification, bounds, entry models.

JEL classification. C14, C35, C71.

1. Introduction

Recent developments in the econometrics of partially identified models have enabled researchers to perform inference in increasingly sophisticated models with strategic interaction. Among them, *static discrete games* have received considerable attention. Some examples include Bjorn and Vuong (1984), Bresnahan and Reiss (1990, 1991), Berry (1992), Tamer (2003), Andrews, Berry, and Jia (2004), Davis (2006), Seim (2006),

Andres Aradillas-Lopez: aaradill@ssc.wisc.edu
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Berry and Tamer (2006), Pakes, Porter, Ho, and Ishii (2006), Pesendorfer and Schmidt-Dengler (2008), Sweeting (2009), Aradillas-Lopez (2010), Galichon and Henry (2011), Beresteanu, Molchanov, and Molinari (2008), Bajari, Hong, Kreiner, and Nekipelov (2010), Bajari, Hong, and Ryan (2010), Ciliberto and Tamer (2009), Kline and Tamer (2011), and de Paula and Tang (2011). A common feature in empirical applications where the space of possible choices (the action space) is large is that the choices observed in the data tend to be heavily concentrated in a small subset of the action space. In fact, some profiles in the action space are never observed in the data. This phenomenon stands in a bit of conflict with the prototypical parametric model, where, in addition to observable covariates, payoffs are assumed to depend on continuously distributed unobservable shocks whose support is unbounded conditional on observables. In those settings, every profile in the action space is the unique equilibrium of the underlying game with nonzero probability and, therefore, every profile in the action space should eventually be observed in the data.

Thus, a natural question to ask is whether the absence of certain action profiles in the data is because equilibria where they are played are rare *or* because such equilibria are systematically avoided by the underlying equilibrium selection mechanism. We say that an action profile $\mathbf{y}$ is a (complete-information) Nash equilibrium (NE) profile if it is played with positive probability in some existing NE. The goal of this paper is to characterize bounds for the probability that a given $\mathbf{y}$ is a NE profile. Comparing the resulting upper bound with the frequency with which $\mathbf{y}$ is the observed outcome of the game, we can also deduce a lower bound for the probability that the equilibrium selection mechanism chooses a NE where $\mathbf{y}$ is played *given that such a NE exists*. Our bounds result from a set of nonparametric assumptions about the underlying payoff functions and on the presupposition that the researcher has ex ante knowledge about the direction of strategic interaction in the game. Due to its fully nonparametric nature and the type of question it addresses, this is the first paper of its kind in the literature on inference in static discrete games.

Being able to isolate (at least partially) the influence of the equilibrium selection mechanism from the structural payoff (equilibrium) properties of the model is a potentially valuable policy tool. For instance, suppose some class $\mathcal{Y}$ of outcomes is deemed "undesirable" for a policy maker. If the model and the data at hand predict that equilibria in $\mathcal{Y}$ exist with very low probability, the policy maker would know that even if the selection mechanism were to change unpredictably, the likelihood of observing outcomes in $\mathcal{Y}$ would remain low. Conversely, if the existence of equilibria in $\mathcal{Y}$ appears to be high, knowing the impact of policy variables on the selection mechanism would be a valuable policy tool.

Robust inference in discrete games is an ongoing area of research. Two recent examples for the case of binary games include Kline and Tamer (2011) and de Paula and Tang (2011). Beyond game-theoretic settings, our model is related to the literature on endogeneity in discrete and limited dependent variable models. A partial list of work in this area includes, among others, Heckman (1978), Sickles and Schmidt (1978), Gourieroux, Laffont, and Montfort (1980), Smith and Blundell (1986), Blundell and Smith (1989, 1994), Sickles (1989), Dagenais (1999), Blundell and Powell (2004), Vytlacil and Yildiz (2007), Lewbel (2007), Chesher (2010), Abrevaya, Hausman, and Khan (2010), and Klein and Vella (2009). Some of these papers focus on a *single* agent making multiple decisions at once. Whether our results can be applied in those settings depends on the appropriateness of our behavioral theory (a non-cooperative game) as a way to represent the behavior of the single agent in question. This is highly debatable and goes beyond the scope of this paper. Methodologically, our main goal is to do robust inference in a problem we consider to be of interest. Robust inference in econometrics is an area of active research. Our approach and methods are particularly motivated by the work of Charles F. Manski and his co-authors. Some examples of work related in some way to this paper include Manski (1990, 1997, 2007), Manski and Pepper (2000), Manski and Tamer (2002), and Imbens and Manski (2004).

The paper proceeds as follows. Section 2 describes the structural and behavioral assumptions of the model. Section 3 details the information available to the researcher. Section 4 derives the observable implications of our assumptions on the properties of Nash equilibria. Equipped with these results, Section 5 derives bounds for the probabilities of interest to us. Section 6 shows how our results simplify in the case of pure-strategy Nash equilibrium. Using the results from Section 5, we describe in Section 7 how to do statistical inference using existing methods. As an empirical illustration, in Section 8 we apply our results to a model of multiple entry by Home Depot and Lowe's in the home improvement retailer industry. Section 9 concludes. Some proofs and extensions to nonequilibrium behavior are included in the Appendixes available in a supplementary file on the journal website, http://qeconomics.org/supp/74/supplement.pdf.

2. Description of the model

2.1 *Agents and actions*

We study a static, simultaneous game played by $p = 1, \dots, P$ agents, each of which has a real-valued decision variable $Y_p \in \mathcal{A}_p \equiv \{0, 1, \dots, M_p\}$ in $\mathbb{N}$, where M_p can be unbounded. We refer to $\mathcal{A}_p$ as p's *action space* and refer to each element in $\mathcal{A}_p$ as an *action*, that is, a potential value for Y_p. We use lowercase (e.g., y_p, v_p, a_p, etc.) to denote actions in $\mathcal{A}_p$. Thus, Y_p is the action chosen by p (i.e., the *choice* made by p). We index p's opponents by $-p$. The space of actions for any subset of players is assumed to be the corresponding Cartesian product of their individual action spaces. In particular, $\mathcal{A}_{-p} \equiv \{(y_q)_{q\neq p} : y_q \in \mathcal{A}_q\}$ denotes the space of actions for p's opponents. We let $\mathcal{A} \equiv \{(y_q)_{q=1}^{P} : y_q \in \mathcal{A}_q\}$ denote the action space for all agents in the model. We use boldface type to denote profiles of actions (and choices) for a subset of players, maintaining the lower- and uppercase distinction between actions and actual choices. Thus, $\mathbf{y}_{-p} \equiv (y_q)_{q\neq p}$ denotes a particular action profile for p's opponents, while $\mathbf{Y}_{-p} \equiv (Y_q)_{q\neq p}$ denotes the profile of actions chosen by p's opponents. A given profile of actions by all agents is denoted, for example, by $\mathbf{y} \equiv (y_p)_{p=1}^{P} \in \mathcal{A}$. Finally, $\mathbf{Y} \equiv (Y_p)_{p=1}^{P}$ (the profile of actions actually chosen by all players in the game) is referred to as the *outcome* of the game.

Remark 1. Uppercase letters Y_p, $\mathbf{Y}_{-p}$, and $\mathbf{Y}$ always denote the actions actually *chosen* by the agents. Conversely, we *always* use lowercase letters y_p, $\mathbf{y}_{-p}$, and $\mathbf{y}$ to denote *given* action profiles in $\mathcal{A}_p$, $\mathcal{A}_{-p}$, and $\mathcal{A}$.

2.2 *Payoffs*

Each agent has a (von Neumann–Morgenstern) payoff function which depends on his own choice, the choices made by his opponents, and a collection of additional payoff shifters. In particular, the payoff to agent p if $Y_p = y_p$ and $\mathbf{Y}_{-p} = \mathbf{y}_{-p}$ is denoted by

$$\nu_p(y_p, \mathbf{y}_{-p}; \omega_p), \tag{1}$$

where ω_p denotes the collection of all other factors (besides Y_p and $\mathbf{Y}_{-p}$) that affect payoffs. Some elements in ω_p may be assumed to be observable to the researcher; however, we do not assume that we can exactly observe or impute ω_p and we do not assume to know its dimension. Note that (1) allows for each p to have a different payoff function. For convenience and to conform to the dimension of $\mathcal{A}_p$, for any $\mathbf{y}_{-p} \in \mathcal{A}_{-p}$, we set $\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) = -\infty$ with probability 1 (w.p.1) for any $y_p < 0$ or $y_p > M_p$. From now on we denote the collection of all players' payoff shifters as $\omega \equiv (\omega_p)_{p=1}^P$.

Strategic substitutability and complementarity. Take any $q \neq p$. We say that Y_q is a strategic *complement* (*substitute*) *of* Y_p if, for almost every realization of ω_p, p's payoff function is nondecreasing (nonincreasing) in Y_q everywhere in $\mathcal{A}$. Otherwise, Y_q is neither a complement nor a substitute for Y_p. This relationship does not have to be symmetric: Y_q could be a strategic substitute for Y_p and Y_p could be a complement for Y_q.

Assumption 1. *With probability* 1 *in* ω_p, *agent p's payoff function satisfies the following conditions*:

(i) Concavity. *Agents' payoffs are strictly concave in their own choice. That is, for any* $\mathbf{y}_{-p} \in \mathcal{A}_{-p}$ *and any* $y_p \in \mathcal{A}_p$,

$$\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p - 1, \mathbf{y}_{-p}; \omega_p) > \nu_p(y_p + 1, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}_{-p}; \omega_p),$$

and if $M_p = \infty$, *then we assume that w.p.*1 *there exists a* $y_p^* \in \mathcal{A}_p$ *such that*

$$\nu_p(y_p^*, \mathbf{y}_{-p}; \omega_p) > \nu_p(y_p^* + 1, \mathbf{y}_{-p}; \omega_p) \quad \forall \mathbf{y}_{-p} \in \mathcal{A}_{-p}.$$

(ii) Local monotonicity. $\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \neq \nu_p(y_p + 1, \mathbf{y}_{-p}; \omega_p)\ \forall y_p \in \mathcal{A}_p, \mathbf{y}_{-p} \in \mathcal{A}_{-p}$.

Assumption 1 ensures[1] the existence of a unique optimal choice for p (in an expected-utility sense) whenever p's opponents are playing a pure strategy. It also ensures that, in any complete-information Nash equilibrium, players can be optimally indifferent between *at most two* actions, which must be *adjacent*.[2]

[1]Notice that while a nonconcave payoff function can be transformed into a concave one by relabeling the actions in $\mathcal{A}_p$, there is no guarantee that the resulting payoff function satisfies Assumption 1(ii) or Assumption 2.

[2]If we focus on pure-strategy NE, we can replace strict concavity with weak concavity as long as local monotonicity is maintained. Strict concavity is key to our observable implications of mixed-strategy NE behavior.

Assumption 2 (Nonincreasing Differences). *For every agent p, all $\mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}$ and $y_p \in \mathcal{A}_p$, the following statement holds with probability* 1:

$$\begin{aligned} &\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \geq \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p) \\ &\implies \quad \nu_p(y_p+1, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \\ &\qquad\qquad \geq \nu_p(y_p+1, \mathbf{y}'_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p). \end{aligned} \tag{2}$$

Given ω_p, take any collection $\mathbf{y}^1_{-p}, \mathbf{y}^2_{-p}, \ldots$ such that $\nu_p(y_p, \mathbf{y}^\ell_{-p}; \omega_p)$ is nonincreasing in ℓ. If (2) holds, the corresponding utility gain from a more aggressive (i.e., larger) choice than y_p is *also nonincreasing* in ℓ. There is an alternative interpretation of (2). Note that it is equivalent to the statement

$$\begin{aligned} &\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \geq \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p) \\ &\implies \quad \nu_p(y_p+1, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p+1, \mathbf{y}'_{-p}; \omega_p) \\ &\qquad\qquad \geq \nu_p(y_p, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p). \end{aligned}$$

That is, Assumption 2 asserts that the more "aggressive" p is, the more susceptible he becomes to the choices made by others. Similar conditions to (2) can be found, for example, in the literature on supermodularity and monotonicity of best-response functions and equilibria; see, for example, Milgrom and Shannon (1994), Echenique and Komunjer (2009), and Section 2.6.1 in Topkis (1998). Related restrictions in the econometric literature can be found, for example, in Davis (2006) and Jia (2008). Figure 1 illustrates the restrictions implied by Assumptions 1 and 2.

Example 1. Consider a market with P firms where Y_p is an indivisible measure of "production" of a good, with $M_p = \infty$ (or arbitrarily large). Suppose that if $Y_p = y_p$ and $\mathbf{Y}_{-p} = \mathbf{y}_{-p}$, the inverse demand function faced by p is $\phi_{1p}(\mathbf{y}_{-p}, \xi^d_p) + \theta_p \cdot y_p$. Demand shifters are $\omega^d_p = (\xi^d_p, \theta_p)$. Suppose firm p's cost function is $\beta^a_p \cdot y_p + \beta^b_p \cdot y^2_p + \xi^c_p$. Cost shifters are $\omega^c_p = (\beta^a_p, \beta^b_p, \xi^c_p)$, and $\omega_p \equiv (\omega^d_p, \omega^c_p)$ denotes all payoff shifters for p. If $Y_p = y_p$ and $\mathbf{Y}_{-p} = \mathbf{y}_{-p}$, firm p's profits are $\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) = [\phi_{1p}(\mathbf{y}_{-p}, \xi^d_p) - \beta^a_p + (\theta_p - \beta^b_p) \cdot y_p] \cdot y_p - \xi^c_p$. Assumption 1 is satisfied if the following conditions hold:

(i) *Concavity.* Payoffs are strictly concave if $\theta^p_p - \beta^b_p < 0$ w.p.1. For this, it would suffice if demand has a negative own price elasticity and marginal costs are nondecreasing w.p.1.

(ii) *Local monotonicity.* A sufficient (but not necessary) condition to satisfy this restriction is if $\phi_{1p}(\mathbf{y}_{-p}, \xi^d_p)$ is continuously distributed conditional on $(\beta^a_p, \theta_p, \beta^b_p)$ for any $\mathbf{y}_{-p} \in \mathcal{A}_{-p}$.

For Assumption 2, note first that for any $y_p \geq 0$ and any realization of ω_p, we have $\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \geq \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p) \iff \phi_{1p}(\mathbf{y}_{-p}, \xi^d_p) \geq \phi_{1p}(\mathbf{y}'_{-p}, \xi^d_p)$. Next note that for

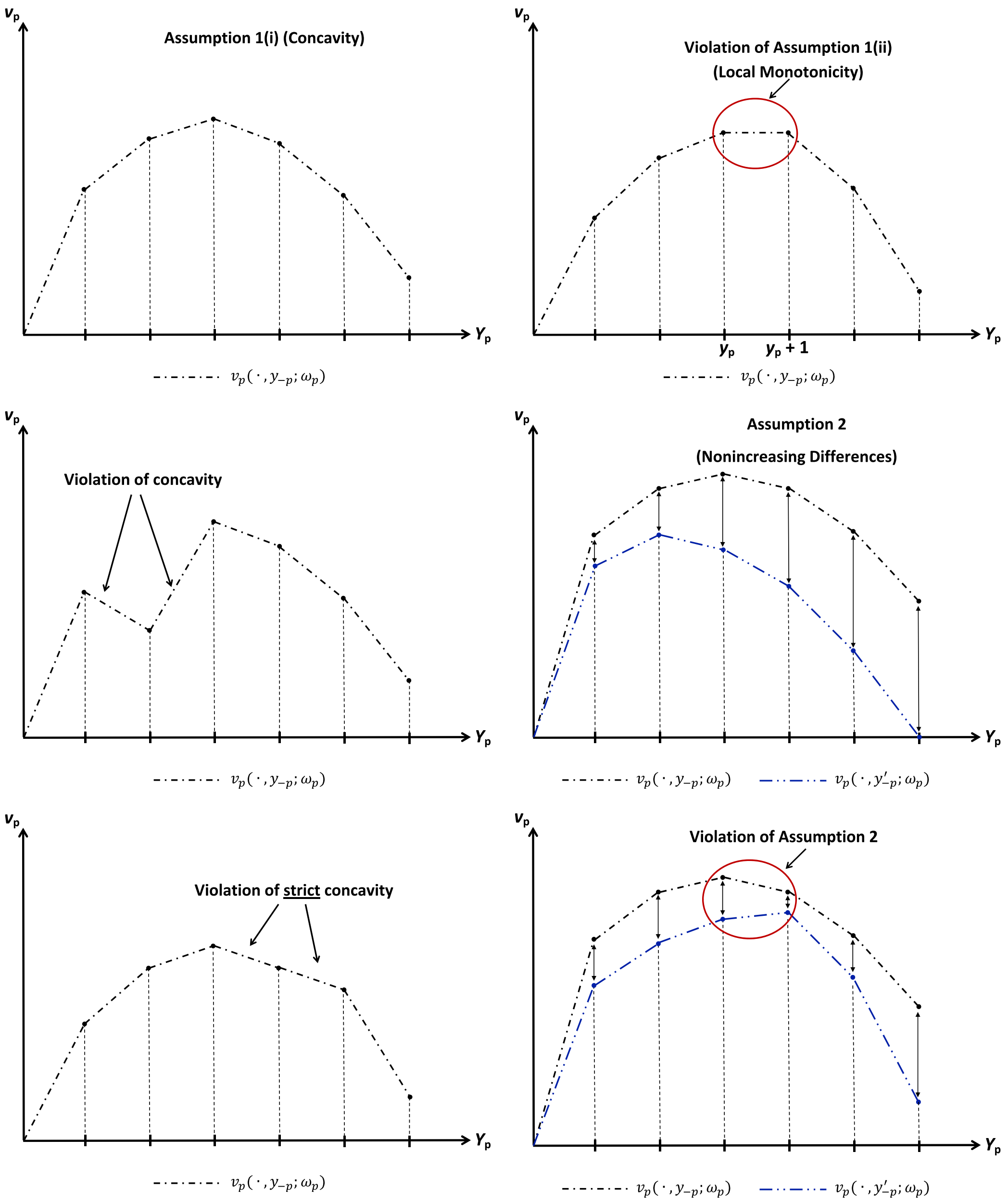

FIGURE 1. Graphical illustration of Assumptions 1 and 2.

any $y_p \geq 0$,

$$\begin{aligned} &\nu_p(y_p + 1, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \\ &\quad = 2 \cdot (\theta_p - \beta_p^b) \cdot y_p + (\theta_p - \beta_p^b - \beta_p^a) + \phi_{1p}(\mathbf{y}_{-p}, \xi_p^d). \end{aligned}$$

Assumption 2 is satisfied since, w.p.1, $\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \geq \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p)$ implies $\nu_p(y_p + 1, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}_{-p}; \omega_p) \geq \nu_p(y_p + 1, \mathbf{y}'_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p)$. Suppose instead that payoffs are of the form

$$\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) = [\phi_{1p}(\mathbf{y}_{-p}, \xi_p^d) - \beta_p^a + (\theta_p - \beta_p^b) \cdot y_p] \cdot y_p - \phi_{2p}(\mathbf{y}_{-p}, \xi_p^c),$$

which arise, for example, if other firms' choices also affect firm p's fixed costs. Assumption 1 is still satisfied under the conditions described above. Assumption 2 imposes some restrictions on the stochastic relationship between $\phi_{1p}(\cdot, \xi_p^d)$ and $\phi_{2p}(\cdot, \xi_p^c)$. More precisely, we need

$$\begin{aligned} &\Pr[\phi_{1p}(\mathbf{y}_{-p}, \xi_p^d) \geq \phi_{1p}(\mathbf{y}'_{-p}, \xi_p^d) | \\ &\qquad \phi_{1p}(\mathbf{y}_{-p}, \xi_p^d) \cdot y_p - \phi_{2p}(\mathbf{y}_{-p}, \xi_p^c) \geq \phi_{1p}(\mathbf{y}'_{-p}, \xi_p^d) \cdot y_p - \phi_{2p}(\mathbf{y}'_{-p}, \xi_p^c)] = 1 \\ &\quad \forall y_p \in \mathcal{A}_p, \forall \mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}. \end{aligned}$$

A *sufficient*, but not necessary condition for this to hold is if

$$\begin{aligned} &\Pr[\phi_{1p}(\mathbf{y}_{-p}, \xi_p^d) \geq \phi_{1p}(\mathbf{y}'_{-p}, \xi_p^d) | \phi_{2p}(\mathbf{y}_{-p}, \xi_p^c) < \phi_{2p}(\mathbf{y}'_{-p}, \xi_p^c)] = 1 \\ &\quad \forall y_p \in \mathcal{A}_p, \forall \mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}, \end{aligned}$$

so that, w.p.1, behavior by opponents which shifts p's costs upward also reduces p's demand.

Remark 2 (Simultaneous Ordered Response Game). The distribution of payoff shifters could be choice-dependent. That is, choosing $Y_p = y_p$ could generate a different draw of ω_p than choosing $Y_p = y'_p$. However, Assumptions 1 and 2 impose significant restrictions on the joint distribution of any pair $\omega_p(y_p)$ and $\omega_p(y'_p)$. In particular, the conditional support of $\omega_p(y_p)$ conditional on $\omega_p(y'_p)$ cannot be unbounded. This rules out, for example, the model described in Equation (1) of Bajari, Hong, and Ryan (2010). Our model should be seen as a *simultaneous ordered response game*, where "ordered response" does not refer in any way to the timing of moves in the game, but rather refers to the econometric usage of the term, referring to a discrete choice model where, for each individual, a *single* index (as opposed to a multivalued index, as is the case in a multinomial choice model) captures the "utility" of each possible choice, and the optimal choice depends on the realization of said index in the real line.

Remark 3. Although our restrictions allow for it, payoff functions are not assumed to change smoothly with ω_p. In particular, even if ω_p is continuous with unbounded support, payoff functions may have a limited number of possible configurations. For instance, for almost every realization of ω_p there could exist an action (which may depend on ω_p) that strictly dominates (expected utilitywise) all other actions in $\mathcal{A}_p$ regardless of the choices made by others. That is, we can have

$$\begin{aligned} &\Pr\bigl[\exists y_p^* : \nu_p(y_p^*, \mathbf{y}_{-p}; \omega_p) > \max\{\nu_p(y_p^* - 1, \mathbf{y}_{-p}; \omega_p), \nu_p(y_p^* + 1, \mathbf{y}_{-p}; \omega_p)\} \\ &\quad \forall \mathbf{y}_{-p} \in \mathcal{A}_{-p}\bigr] = 1. \end{aligned} \tag{3}$$

We can also have actions in $\mathcal{A}_p$ that are strictly dominated w.p.1. Thus, our assumptions are consistent with data where some action profiles in $\mathcal{A}$ are never observed as the outcome of the game. This is typically not the case for parametric models with continuously distributed unobservable payoff shocks with unbounded support.

The following consequence of nonincreasing differences is very useful to us.

RESULT 1 (Payoff Functions Do Not Cross). *If Assumption 2 holds, then payoff functions almost surely do not cross. That is, for almost every realization of* ω_p, *we have*

$$\forall \mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}, \nexists y_p, y'_p \in \mathcal{A}_p \text{ such that}$$
$$\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) < \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p) \quad \text{and} \quad \nu_p(y'_p, \mathbf{y}_{-p}; \omega_p) > \nu_p(y'_p, \mathbf{y}'_{-p}; \omega_p).$$

Therefore, for any $\mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}$, *w.p.*1, *we have either* $\nu_p(\cdot, \mathbf{y}_{-p}; \omega_p) \geq \nu_p(\cdot, \mathbf{y}'_{-p}; \omega_p)$ *or* $\nu_p(\cdot, \mathbf{y}_{-p}; \omega_p) \leq \nu_p(\cdot, \mathbf{y}'_{-p}; \omega_p)$.

PROOF. A violation of Result 1 implies the existence of a $y_p \in \mathcal{A}_p$ such that $\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) < \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p)$ and $\nu_p(y_p + 1, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}_{-p}; \omega_p) < \nu_p(y_p + 1, \mathbf{y}'_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p)$, but this violates the nonincreasing differences condition in Assumption 2. □

2.3 *Behavior: Complete-information Nash equilibrium*

Agents make their choices simultaneously in a *complete-information* environment where the realization of payoff functions is publicly observed. A *mixing strategy* by agent p is a probability function $\pi_p : \mathcal{A}_p \longrightarrow [0, 1]$. Recall that $\omega \equiv (\omega_p)_{p=1}^P$ denotes the collection of all agents' payoff shifters. A mixing strategy by p's opponents is a probability function over $\mathcal{A}_{-p}$, and we denote it by π_{-p}. In our setting, for a given ω, a *Nash equilibrium* is a probability function $\pi : \mathcal{A} \longrightarrow [0, 1]$ induced by a collection of mixing strategies $(\pi_p)_{p=1}^P$ through independent randomization. That is, $\pi(\mathbf{y}) = \prod_{p=1}^P \pi_p(y_p)$ for each $\mathbf{y} \equiv (y_p)_{p=1}^P \in \mathcal{A}$, and $\pi_{-p}(\mathbf{y}_{-p}) = \prod_{q \neq p} \pi_q(y_q)$ for each $\mathbf{y}_{-p} \equiv (y_q)_{q \neq p} \in \mathcal{A}_{-p}$ and each p. In addition, for each p,

$$y_p \in \underset{y \in \mathcal{A}_p}{\arg\max} \left\{ \sum_{\mathbf{y}_{-p} \in \mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}) \cdot \nu_p(y, \mathbf{y}_{-p}; \omega_p) \right\} \quad \forall y_p : \pi_p(y_p) > 0.$$

That is, every action chosen with positive probability by π_p is expected-utility maximizing given the mixing strategies of p's opponents.

Nash equilibrium profile. We say that $\mathbf{y} \in \mathcal{A}$ is a Nash equilibrium profile if there exists a NE π such that $\pi(\mathbf{y}) > 0$; that is, a NE where $\mathbf{y}$ is played with positive probability. We define

$$\mathscr{E}(\omega) = \{\mathbf{y} \in \mathcal{A} : \mathbf{y} \text{ is a NE profile}\}.$$

We assume $\mathscr{E}(\omega)$ to be nonempty w.p.1.

Assumption 3. *The outcome of the game is a NE profile w.p.*1; *that is,* $\mathbf{Y} \in \mathscr{E}(\omega)$ *w.p.*1. *More precisely, there exists a* (*possibly random*) *mechanism* $\mathcal{M}_{\mathscr{E}}$ *that, for each* ω, *selects a NE* π^* *from the set of existing NE, and the outcome observed* $\mathbf{Y}$ *is the realization of this NE*; *therefore,* $\pi^*(\mathbf{Y}) > 0$. *Unless we explicitly say otherwise, the features of the equilibrium selection mechanism* $\mathcal{M}_{\mathscr{E}}$ *are left completely unspecified.*

Our goal is to characterize bounds for $\Pr[\mathbf{y} \in \mathscr{E}(\omega)|X]$ for any given $\mathbf{y}$ in $\mathcal{A}$, where X are observable payoff covariates. By comparing these bounds with $\Pr[\mathbf{Y} = \mathbf{y}|X]$, we also obtain a lower bound for $\Pr[\mathcal{M}_{\mathscr{E}}$ selects a NE $\pi : \pi(\mathbf{y}) > 0|\mathbf{y} \in \mathscr{E}(\omega), X]$. While maintaining Assumptions 1–3, we also presuppose that the researcher has ex ante knowledge about the direction of strategic interaction. We describe this in the next section.

3. Information available to the researcher

We now describe the inferential setting faced by the researcher.

3.1 *Features of a prototypical data set*

For a sample of $i = 1, \dots, N$ games which satisfy our assumptions, the researcher observes the corresponding outcomes $\mathbf{Y}_i \equiv (Y_{p,i})_{p=1}^P$ and (possibly) a vector of covariates X_i, which may be a subset of $\omega_i \equiv (\omega_{p,i})_{p=1}^P$ or, more generally, may be related to ω_i. The dimension of the elements in ω_i that are not controlled by X_i is unspecified. The researcher cannot determine whether, for any pair of observations $i \neq j$, the corresponding outcomes $\mathbf{Y}_i$ and $\mathbf{Y}_j$ were produced by the same realization of payoff functions. We rule out, for instance, the availability of panel data where some prespecified feature of payoff functions is assumed to be fixed across some dimension of the panel. All of our results could be refined in such a setting. In our statistical inference section, we assume $(\mathbf{Y}_i, X_i)_{i=1}^N$ to be an independent and identically distributed (iid) sample, but our final results can potentially be modified to allow for dependence across observations.

The labels of players $p = 1, \dots, P$ can be meaningful in various ways, depending on the application at hand. For instance, $p = 1$ and $p = 2$ could denote specific firms (e.g., Home Depot and Lowe's), and thus $Y_{1,i}$ and $Y_{2,i}$ would denote the action chosen by each of these firms in the ith market. Alternatively, $p = 1$ and $p = 2$ could refer to the largest and the second largest firms in a market; to the incumbent and the entrant, and so on. We could also have applications where the game is played by P symmetric players, where their labeling would be unnecessary.

3.2 *Ex ante knowledge about the direction of strategic interaction*

The functional forms of payoffs are unknown. In this setting, we obtain constructive results by assuming the existence of ex ante knowledge about the *direction* of strategic interaction in the game. To be precise, for any given $\mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}$, we assume that the researcher knows *sufficient* conditions such that $\nu_p(\cdot, \mathbf{y}_{-p}; \omega_p) \geq \nu_p(\cdot, \mathbf{y}'_{-p}; \omega_p)$ w.p.1. This knowledge may come from economic theory or it could be a modeling assumption maintained by the researcher. For instance, if economic theory predicts that Y_p

and Y_q are pairwise substitutes w.p.1, then having $\mathbf{y}'_{-p} \geq \mathbf{y}_{-p}$ (elementwise) would imply $\nu_p(\cdot, \mathbf{y}'_{-p}; \omega_p) \leq \nu_p(\cdot, \mathbf{y}_{-p}; \omega_p)$ w.p.1. We formalize our assumption next.

Assumption 4. *For each p there exists a function $f_p : \mathcal{A}_{-p} \to \mathbb{R}^{d_p}$ (with $d_p \geq 1$) known to the researcher such that the following conditions hold*:

(i) *For all $\mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}$, $f_p(\mathbf{y}'_{-p}) \geq f_p(\mathbf{y}_{-p})$ (elementwise) $\Longrightarrow \nu_p(\cdot, \mathbf{y}'_{-p}; \omega_p) \leq \nu_p(\cdot, \mathbf{y}_{-p}; \omega_p)$ w.p.1.*

(ii) *If f_p is multivalued, and we have both $f_p(\mathbf{y}'_{-p}) \ngeq f_p(\mathbf{y}_{-p})$ and $f_p(\mathbf{y}_{-p}) \ngeq f_p(\mathbf{y}'_{-p})$, it is impossible for the researcher to predict, conditional on observables, the ordinal relationship between $\nu_p(y_p, \mathbf{y}'_{-p}; \omega_p)$ and $\nu_p(y_p, \mathbf{y}_{-p}; \omega_p)$ for any y_p.*

With probability 1, having $f_p(\mathbf{y}'_{-p}) \geq f_p(\mathbf{y}_{-p})$ ensures that[3], for any possible choice p can make, his corresponding payoff if $\mathbf{Y}_{-p} = \mathbf{y}_{-p}$ cannot be smaller than he would obtain if $\mathbf{Y}_{-p} = \mathbf{y}'_{-p}$. If f_p is multivalued, we cannot conclude anything if $f_p(\mathbf{y}'_{-p}) \ngeq f_p(\mathbf{y}_{-p})$ and $f_p(\mathbf{y}_{-p}) \ngeq f_p(\mathbf{y}'_{-p})$. For almost every realization of ω_p and any action p can choose, opponent action profiles in $\mathcal{A}_{-p}$ that yield higher (elementwise) values of f_p cannot leave p better off.

Example 1 (Continued). We revisit the example from Section 2 and outline two instances of the type of maintained presuppositions that lead to the setting described in Assumption 4.

(i) Suppose every other Y_q can be classified as either a complement or a substitute of Y_p w.p.1, and that the researcher knows the identities of each group. In addition, it is maintained that p's payoffs depend on others' actions only through *the total quantity produced by each group of opponents*. Assumption 4 is then satisfied by the function $f_p : \mathcal{A}_{-p} \longrightarrow \mathbb{R}^2$ given by

$$f_p(\mathbf{y}_{-p}) = \left(\sum_{q \in \mathscr{S}_p} y_q, -\sum_{q \in \mathscr{C}_p} y_q \right),$$

where $\mathscr{S}_p$ and $\mathscr{C}_p$ denote the group of substitutes and complements of Y_p.

(ii) Again, suppose every other Y_q can be classified as either a complement or a substitute of Y_p w.p.1, and that the researcher knows the identities of each group. However, nothing else is assumed about how p's payoffs depend on others' actions. Assumption 4 is then satisfied by the function $f_p : \mathcal{A}_{-p} \longrightarrow \mathbb{R}^{P-1}$ given by

$$f_p(\mathbf{y}_{-p}) = \big((y_q)_{q \in \mathscr{S}_p}, (-y_q)_{q \in \mathscr{C}_p} \big).$$

Again, $\mathscr{S}_p$ and $\mathscr{C}_p$ denote the group of substitutes and complements of Y_p.

[3] Since payoff functions do not cross (see Result 1), it would be enough to state Assumption 4(i) as "$\forall \mathbf{y}_{-p}, \mathbf{y}'_{-p} \in \mathcal{A}_{-p}, f_p(\mathbf{y}'_{-p}) \geq f_p(\mathbf{y}_{-p})$ (elementwise) $\Longrightarrow \nu_p(y_p, \mathbf{y}'_{-p}; \omega_p) \leq \nu_p(y_p, \mathbf{y}_{-p}; \omega_p)$ for some y_p."

Applications where Assumption 4 does not hold are not suitable for our methods. Furthermore, without stronger conditions than those described in Assumptions 1–3, we cannot test Assumption 4, so it must be a maintained restriction. Maintaining ex ante knowledge of the direction of strategic interaction is a commonly found identification assumption in parametric models of complete-information games. Some examples include Bresnahan and Reiss (1990, 1991), Berry (1992), Tamer (2003), and Davis (2006). As Example 1 (in particular part (ii)) illustrates, even though it implicitly adds more structure, Assumption 4 does not necessarily amount to the parametrization of payoff functions.

4. Implications of our assumptions on the properties of Nash equilibria

4.1 *Implications of Assumptions 1 and 2*

The first implication of our payoff assumptions follows from the independent mixing in NE.

4.1.1 *Features of the support of Nash equilibria*

Result 2. *If Assumption 1*(i) *holds* (*strict concavity*), *then in any NE, each agent can play at most two actions with positive probability, and these actions must be adjacent. Furthermore, a direct consequence of Assumption 1*(ii) *is that there almost surely cannot exist a NE where only one agent is playing a mixed strategy.*

Proofs for Result 2 and Propositions 1, 2, and 4 are provided in Appendix A.

Once ω is realized, independent mixing in NE implies that the expected utility of choosing $Y_p = y_p$ is always of the form $\sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p})\cdot\nu_p(y_p, \mathbf{y}_{-p}; \omega_p)$, as opposed to the more general

$$\sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}|y_p)\cdot\nu_p(y_p, \mathbf{y}_{-p}; \omega_p),$$

where p's opponents' mixing distribution can depend on the action chosen by p. For any well defined probability function $\pi_{-p}:\mathcal{A}_{-p}\longrightarrow[0,1]$, strict concavity of payoffs (Assumption 1(i)) yields

$$\begin{aligned}&\sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p})\cdot[\nu_p(y_p, \mathbf{y}_{-p}; \omega_p)-\nu_p(y_p-1, \mathbf{y}_{-p}; \omega_p)]\\&\quad>\sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p})\cdot[\nu_p(y_p+1, \mathbf{y}_{-p}; \omega_p)-\nu_p(y_p, \mathbf{y}_{-p}; \omega_p)]\end{aligned}$$

for any $y_p\in\mathcal{A}_p$. It follows that in any NE π, agent p can only be optimally indifferent between at most two actions, which must be adjacent. Coordination across agents (e.g., as in a correlated equilibrium[4]) and, in general, any departure from independent mixing of

[4] See Definition 2.4B in Fudenberg and Tirole (1991).

NE could invalidate Result 2. *Failure of strict concavity of payoffs could also invalidate it.* With weak concavity (see the lower left panel in Figure 1), agents can be made optimally indifferent across three or more actions. Private information in payoffs could, in general, also bring down Result 2, even if strict concavity holds and Bayesian–Nash equilibrium behavior prevails.

A generic characterization of the support of Nash equilibria. Without imposing stronger conditions, the only constructive implications of our assumptions on the features of NE distributions are the adjacent-action support restrictions described in Result 2. From there, we can express the support of *any* NE as a Cartesian product of the form

$$\mathcal{S}=\prod_{p=1}^{P}\{a_p,b_p\}=\{(y_p)_{p=1}^{P}:y_p=a_p \text{ or } y_p=b_p \text{ for each } p\}, \tag{4}$$

where

(i) $a_p, b_p \in \mathcal{A}_p \ \forall p$,

(ii) $b_p = a_p$ or $b_p = a_p + 1 \ \forall p$,

(iii) if $a_p \neq b_p$ for some p, then $a_q \neq b_q$ for some $q \neq p$.

We should think about a_p and b_p as the two actions that player p is supposed to play with positive probability in a hypothetical NE (if $a_p = b_p$, then p is supposed to play a_p as a pure strategy), and we should think of $\mathcal{S}$ as the support of said NE. For example, suppose $P = 2$ (two players) and take any $\mathbf{y} \equiv (y_1, y_2)$ in the interior of $\mathcal{A}$. Then the support $\mathcal{S}$ of any NE where $\mathbf{y}$ is played with positive probability must be one of the five sets

$$\begin{aligned}&\{y_1-1,y_1\}\times\{y_2-1,y_2\}, && \{y_1,y_1+1\}\times\{y_2-1,y_2\},\\ &\{y_1-1,y_1\}\times\{y_2,y_2+1\}, && \{y_1,y_1+1\}\times\{y_2,y_2+1\}, \quad \text{or} \quad \{y_1,y_1\}\times\{y_2,y_2\}.\end{aligned}$$

For instance, the first case requires agent 1 to randomize between $y_1 - 1$ and y_1, and requires agent 2 to randomize between $y_2 - 1$ and y_2. The last case corresponds to playing y_1 and y_2 as pure strategies. As usual, for any p and $\mathbf{y}_{-p} \equiv (y_q)_{q\neq p}$, we say that $\mathbf{y}_{-p} \in \mathcal{S}$ if there exists a profile $\mathbf{v} \equiv (v_p)_{p=1}^{P}$ in $\mathcal{S}$ where $v_q = y_q$ for each $q \neq p$.

How large is the collection of *different* sets $\mathcal{S}$ that can constitute the support of a NE where $\mathbf{y}$ is played with positive probability? A simple counting exercise shows that

$$\#\{\mathcal{S}:\mathbf{y}\in\mathcal{S} \text{ and } \mathcal{S} \text{ satisfies (4)}\}\leq\sum_{R=2}^{P}\binom{P}{R}\cdot 2^R+1=3^P-2P. \tag{5}$$

This bound holds exactly if $\mathbf{y}$ belongs in the interior of $\mathcal{A}$ and is a strict inequality otherwise. For instance, if $P = 2$, this bound is 5 (see the above example) and it is 21 if $P = 3$.

4.1.2 *Necessary conditions for the existence of NE with a prespecified support* We begin by exploring the implications of Assumptions 1 and 2 on the existence of NE with a prespecified support $\mathcal{S}$. Afterward, we move on to necessary conditions for the coexistence of a NE with support $\mathcal{S}$ and a NE with support $\mathcal{S}'$.

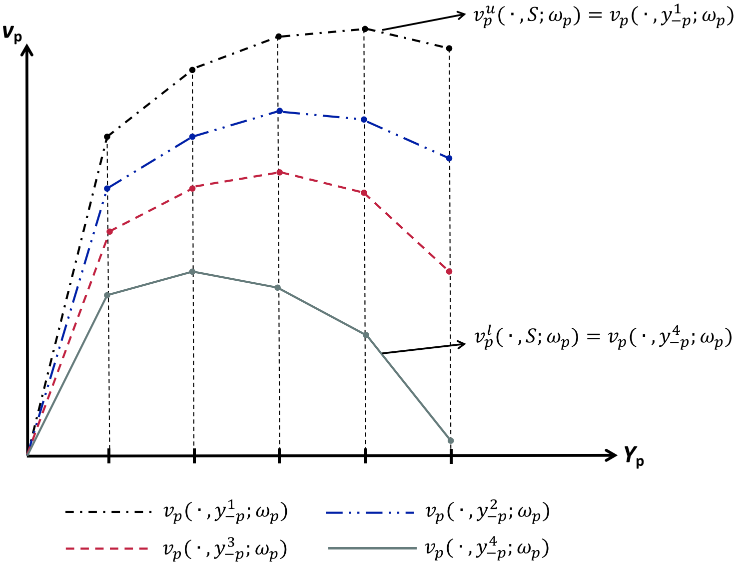

FIGURE 2. $\nu_p^u(\cdot, \mathcal{S}; \omega_p)$ and $\nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$ for a collection $\mathcal{S}$ that includes $\mathbf{y}_{-p}^1$, $\mathbf{y}_{-p}^2$, $\mathbf{y}_{-p}^3$, and $\mathbf{y}_{-p}^4$.

DEFINITION 1. ($\nu_{\mathbf{p}}^{\mathbf{u}}(\cdot, \mathcal{S}; \omega_{\mathbf{p}}), \nu_{\mathbf{p}}^\ell(\cdot, \mathcal{S}; \omega_{\mathbf{p}})$). Let $\mathcal{S}$ be a set as described in (4). For any such set, we define

$$\begin{aligned} &\nu_p^u(\cdot, \mathcal{S}; \omega_p) = \max\{\nu_p(\cdot, \mathbf{y}_{-p}; \omega_p) : \mathbf{y}_{-p} \in \mathcal{S}\} \quad \text{and} \\ &\nu_p^\ell(\cdot, \mathcal{S}; \omega_p) = \min\{\nu_p(\cdot, \mathbf{y}_{-p}; \omega_p) : \mathbf{y}_{-p} \in \mathcal{S}\}. \end{aligned} \tag{6}$$

Once ω_p is realized, $\nu_p^u(\cdot, \mathcal{S}; \omega_p)$ and $\nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$ simply denote upper and lower bounds (envelopes) for p's payoff function whenever p's opponents choose an action profile in $\mathcal{S}$. Figure 2 illustrates these objects for a hypothetical $\mathcal{S}$ which includes four action profiles $\{\mathbf{y}_{-p}^\ell\}_{\ell=1}^4$.

As a consequence of the no-crossing property of payoff functions shown in Result 1, we can see in Figure 2 that for the underlying realization of ω_p depicted there, we had $\nu_p^u(\cdot, \mathcal{S}; \omega_p) = \nu_p(\cdot, \mathbf{y}_{-p}^1; \omega_p)$ and $\nu_p^\ell(\cdot, \mathcal{S}; \omega_p) = \nu_p(\cdot, \mathbf{y}_{-p}^4; \omega_p)$. This is actually a general feature of $\nu_p^u(\cdot, \mathcal{S}; \omega_p)$ and $\nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$ for any $\mathcal{S}$. The no-crossing property of payoffs implies that for any $\mathcal{S}$, once ω_p is realized, there exist $\mathbf{y}_{-p}^* \in \mathcal{S}$ and $\mathbf{y}_{-p}^{**} \in \mathcal{S}$ (not necessarily unique) such that

$$\nu_p^u(\cdot, \mathcal{S}; \omega_p) = \nu_p(\cdot, \mathbf{y}_{-p}^*; \omega_p) \quad \text{and} \quad \nu_p^\ell(\cdot, \mathcal{S}; \omega_p) = \nu_p(\cdot, \mathbf{y}_{-p}^{**}; \omega_p).$$

As a consequence, *both* $\nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$ *and* $\nu_p^u(\cdot, \mathcal{S}; \omega_p)$ *must satisfy the shape restrictions in Assumptions* 1 *and* 2. In particular, in addition to having to satisfy concavity, nonincreasing differences (Assumption 2) implies that for any finite collection $\mathcal{S}$ and *each* y_p in $\mathcal{A}$, w.p.1, we must have

$$\nu_p^u(y_p + 1, \mathcal{S}; \omega_p) - \nu_p^u(y_p, \mathcal{S}; \omega_p) \geq \nu_p^\ell(y_p + 1, \mathcal{S}; \omega_p) - \nu_p^\ell(y_p, \mathcal{S}; \omega_p),$$

$$
\begin{aligned}
&\nu_p^u(y_p+1,\mathcal{S};\omega_p)-\nu_p^u(y_p,\mathcal{S};\omega_p)\\
&\quad\geq \nu_p(y_p+1,\mathbf{y}_{-p};\omega_p)-\nu_p(y_p,\mathbf{y}_{-p};\omega_p)\quad \forall \mathbf{y}_{-p}\in\mathcal{S},\\
&\nu_p(y_p+1,\mathbf{y}_{-p};\omega_p)-\nu_p^\ell(y_p,\mathbf{y}_{-p};\omega_p)\\
&\quad\geq \nu_p^\ell(y_p+1,\mathcal{S};\omega_p)-\nu_p(y_p,\mathcal{S};\omega_p)\quad \forall \mathbf{y}_{-p}\in\mathcal{S},
\end{aligned}
\tag{7}
$$

and w.p.1, if the collections $\mathcal{S}$ and $\mathcal{S}'$ are such that $\nu_p^\ell(\cdot,\mathcal{S}';\omega_p)\geq\nu_p^u(\cdot,\mathcal{S};\omega_p)$, we must also have

$$
\nu_p^\ell(y_p+1,\mathcal{S}';\omega_p)-\nu_p^\ell(y_p,\mathcal{S}';\omega_p)\geq\nu_p^u(y_p+1,\mathcal{S};\omega_p)-\nu_p^u(y_p,\mathcal{S};\omega_p). \tag{8}
$$

Proposition 1. *Let $\mathcal{S}$ be as described in* (4). *If Assumptions* 1 *and* 2 *hold, there exists a NE π with support $\mathcal{S}$ only if the following two conditions hold*:

(i) $\nu_p^u(a_p,\mathcal{S};\omega_p)<\nu_p^u(a_p+1,\mathcal{S};\omega_p)$ *and* $\nu_p^\ell(a_p,\mathcal{S};\omega_p)>\nu_p^\ell(a_p+1,\mathcal{S};\omega_p)\ \forall p: b_p=a_p+1$.

(ii) $\nu_p^u(a_p-1,\mathcal{S};\omega_p)<\nu_p^u(a_p,\mathcal{S};\omega_p)$ *and* $\nu_p^\ell(a_p,\mathcal{S};\omega_p)>\nu_p^\ell(a_p+1,\mathcal{S};\omega_p)\ \forall p: b_p=a_p$.

Proposition 1 follows from the shape restrictions in Assumptions 1 and 2 and their implication in (7). The details are in the Appendices, but Figure 3 illustrates the arguments. If $b_p=a_p+1$, any NE with support $\mathcal{S}$ requires p to randomize across a_p and a_p+1. Panel (A) depicts the restrictions implied by part (i) of Proposition 1 along with an illustration of how, if these restrictions are satisfied, we may find a mixing distribution π_{-p} with support $\mathcal{S}$ that makes p optimally indifferent between a_p and a_p+1. Panels (B) and (C) show that if Assumptions 1 and 2 hold and if the conditions in part (i) are violated, there cannot be such mixing distribution. Panel (D) depicts the restrictions stated in part (ii) of Proposition 1; that is, the restrictions needed to play $Y_p=a_p$ as a pure-strategy in a NE with support $\mathcal{S}$. Panels (E) and (F) show how, if these conditions are violated, our payoff assumptions imply that $Y_p=a_p$ cannot be an optimal choice for *any* mixing distribution π_{-p} with support $\mathcal{S}$.

4.1.3 *Necessary conditions for the coexistence of Nash equilibria with prespecified supports* Let $\mathcal{S}$ and $\mathcal{S}'$ be any pair of sets that satisfy the restrictions in (4). Maintaining that a NE with support $\mathcal{S}$ exists, we use the results from Proposition 1 and Assumptions 1 and 2 to characterize necessary conditions for the coexistence of a NE with support $\mathcal{S}'$.

Proposition 2. *Let $\mathcal{S}=\prod_{p=1}^P\{a_p,b_p\}$ and $\mathcal{S}'=\prod_{p=1}^P\{a'_p,b'_p\}$ satisfy the conditions in* (4). *Suppose there exists a NE π with support $\mathcal{S}$. If Assumptions* 1 *and* 2 *hold, a NE π' with support $\mathcal{S}'$ also exists only if, for each p, either $\nu_p^\ell(\cdot,\mathcal{S}';\omega_p)<\nu_p^u(\cdot,\mathcal{S};\omega_p)$ and $\nu_p^u(\cdot,\mathcal{S}';\omega_p)>\nu_p^\ell(\cdot,\mathcal{S};\omega_p)$ or one of the following cases holds*:

Case I. *If $a_p\neq b_p$ and $a'_p\neq b'_p$* (*i.e.*, $b_p=a_p+1$ *and* $b'_p=a'_p+1$), *there are two alternatives*:

(i) *If* $\nu_p^\ell(\cdot,\mathcal{S}';\omega_p)\geq\nu_p^u(\cdot,\mathcal{S};\omega_p)$, *we must have* $a'_p>a_p$.

(ii) *If* $\nu_p^u(\cdot,\mathcal{S}';\omega_p)\leq\nu_p^\ell(\cdot,\mathcal{S};\omega_p)$, *we must have* $a'_p<a_p$.

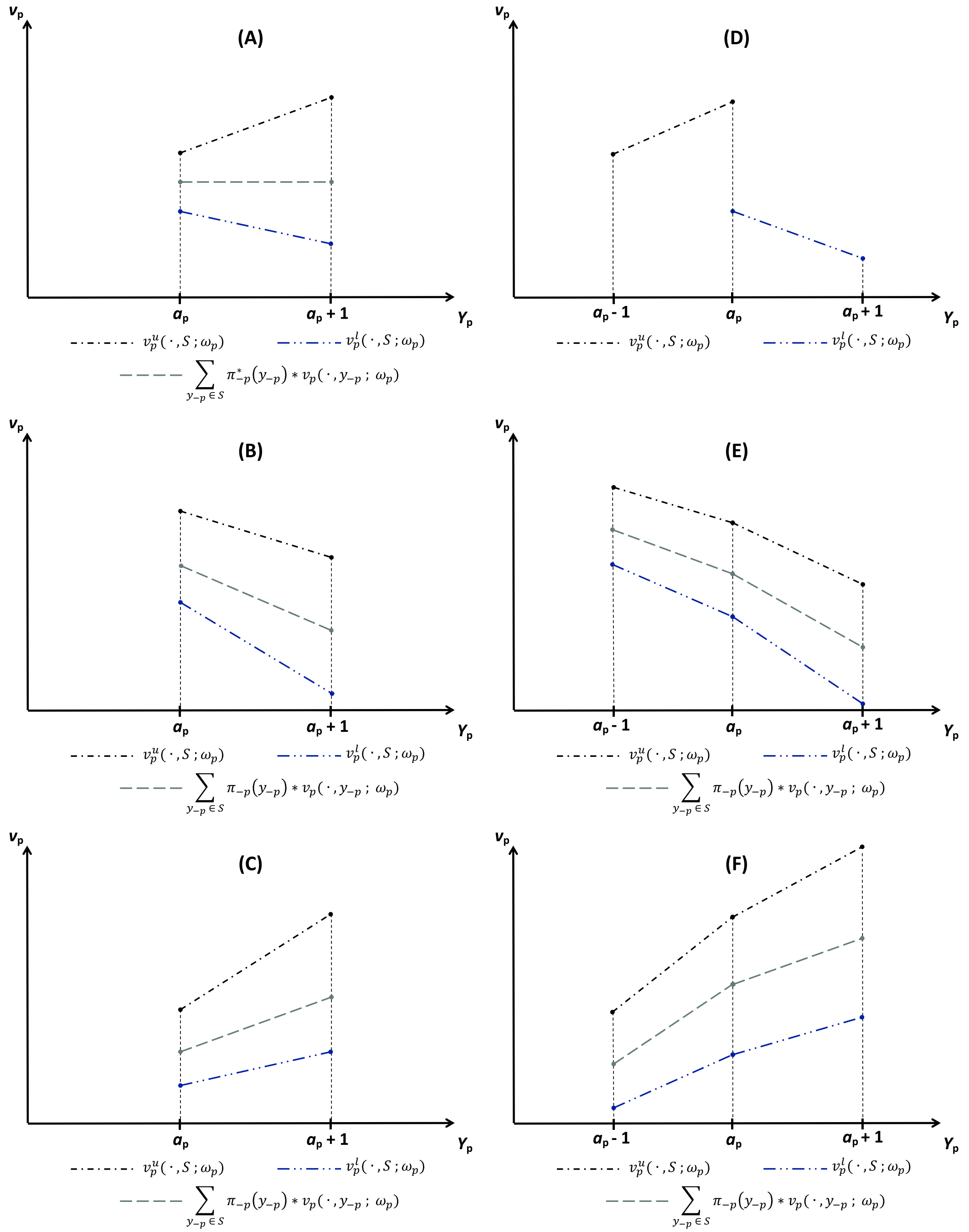

FIGURE 3. Illustration of Proposition 1. Panel (A) depicts the restrictions in part (i), while (B) and (C) show the implications of their violation. Panel (D) describes the restrictions in part (ii), while (E) and (F) depict the implications of their violation.

Case II. *If* $a_p \neq b_p$ (*i.e.*, $b_p = a_p + 1$) *and* $a'_p = b'_p$, *there are two alternatives*:

(i) *If* $\nu^{\ell}_p(\cdot, \mathcal{S}'; \omega_p) \geq \nu^u_p(\cdot, \mathcal{S}; \omega_p)$, *we must have* $a'_p > a_p$.

(ii) *If* $\nu^u_p(\cdot, \mathcal{S}'; \omega_p) \leq \nu^{\ell}_p(\cdot, \mathcal{S}; \omega_p)$, *we must have* $a'_p \leq a_p$.

Case III. *If* $a_p = b_p$ *and* $a'_p \neq b'_p$ (*i.e.*, $b'_p = a'_p + 1$), *there are two alternatives*:

(i) *If* $\nu^{\ell}_p(\cdot, \mathcal{S}'; \omega_p) \geq \nu^u_p(\cdot, \mathcal{S}; \omega_p)$, *we must have* $a'_p \geq a_p$.

(ii) *If* $\nu^u_p(\cdot, \mathcal{S}'; \omega_p) \leq \nu^{\ell}_p(\cdot, \mathcal{S}; \omega_p)$, *we must have* $a'_p < a_p$.

Case IV. *If* $a_p = b_p$ *and* $a'_p = b'_p$, *there are two alternatives*:

(i) *If* $\nu^{\ell}_p(\cdot, \mathcal{S}'; \omega_p) \geq \nu^u_p(\cdot, \mathcal{S}; \omega_p)$, *we must have* $a'_p \geq a_p$.

(ii) *If* $\nu^u_p(\cdot, \mathcal{S}'; \omega_p) \leq \nu^{\ell}_p(\cdot, \mathcal{S}; \omega_p)$, *we must have* $a'_p \leq a_p$.

If the above conditions are satisfied, then the payoff restrictions necessary for the coexistence of NE with supports $\mathcal{S}$ *and* $\mathcal{S}'$ *are compatible with Assumptions* 1 *and* 2.

Proposition 2 follows from the results in Proposition 1 and the payoff restrictions implied by Assumptions 1 and 2; in particular, the implications described in (8). Figure 4 illustrates Case I(i), where p is required to randomize between a_p and $a_p + 1$ in the first NE, and between a'_p and $a'_p + 1$ in the second NE. From part (i) of Proposition 1, we must have $\nu^u_p(a_p, \mathcal{S}; \omega_p) < \nu^u_p(a_p + 1, \mathcal{S}; \omega_p)$ and $\nu^{\ell}_p(a'_p, \mathcal{S}'; \omega_p) > \nu^{\ell}_p(a'_p + 1, \mathcal{S}'; \omega_p)$. If $\nu^{\ell}_p(\cdot, \mathcal{S}'; \omega_p) \geq \nu^u_p(\cdot, \mathcal{S}; \omega_p)$ and payoffs are concave, then having $a'_p \leq a_p$ implies a violation of Assumption 2 (specifically, of Equation (8)). All other cases in Proposition 2 are

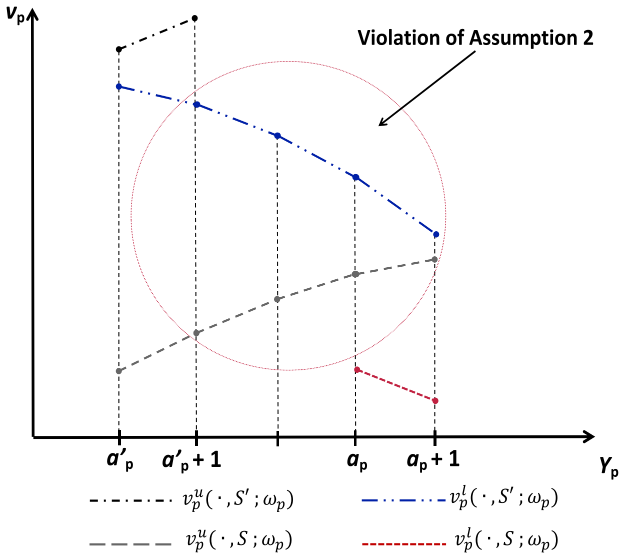

FIGURE 4. By concavity of payoffs, a violation of the restriction in Case I(i) of Proposition 2 necessarily implies a violation of Assumption 2 (specifically, a violation of Equation (8)).

established analogously; please see the details in Appendix A.

For notational convenience, we use an indicator function for the event that the conditions in Proposition 2 are satisfied for a given pair $(\mathcal{S}, \mathcal{S}')$. We denote it by $\mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega)$ and we describe it next.

Definition 2 ($\mathbb{I}_{\mathbf{p}}(\mathcal{S}, \mathcal{S}'; \omega_{\mathbf{p}}), \mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega)$). Let $\mathcal{S} = \prod_{p=1}^{P}\{a_p, b_p\}$ and $\mathcal{S}' = \prod_{p=1}^{P}\{a'_p, b'_p\}$ satisfy the conditions in (4). For each p, let $\mathbb{I}_p(\mathcal{S}, \mathcal{S}'; \omega_p)$ be an indicator function defined as follows.

Case I. If $a_p \neq b_p$ and $a'_p \neq b'_p$ (i.e., $b_p = a_p + 1$ and $b'_p = a'_p + 1$), let

$$\begin{aligned}\mathbb{I}_p(\mathcal{S}, \mathcal{S}'; \omega_p) = 1 - \max\bigl\{&\mathbb{1}\{\nu_p^{\ell}(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^{u}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p \leq a_p\},\\ &\mathbb{1}\{\nu_p^{u}(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^{\ell}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p \geq a_p\}\bigr\}.\end{aligned}$$

Case II. If $a_p \neq b_p$ (i.e., $b_p = a_p + 1$) and $a'_p = b'_p$, let

$$\begin{aligned}\mathbb{I}_p(\mathcal{S}, \mathcal{S}'; \omega_p) = 1 - \max\bigl\{&\mathbb{1}\{\nu_p^{\ell}(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^{u}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p \leq a_p\},\\ &\mathbb{1}\{\nu_p^{u}(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^{\ell}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p > a_p\}\bigr\}.\end{aligned}$$

Case III. If $a_p = b_p$ and $a'_p \neq b'_p$ (i.e., $b'_p = a'_p + 1$), let

$$\begin{aligned}\mathbb{I}_p(\mathcal{S}, \mathcal{S}'; \omega_p) = 1 - \max\bigl\{&\mathbb{1}\{\nu_p^{\ell}(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^{u}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p < a_p\},\\ &\mathbb{1}\{\nu_p^{u}(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^{\ell}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p \geq a_p\}\bigr\}.\end{aligned}$$

Case IV. If $a_p = b_p$ and $a'_p = b'_p$, let

$$\begin{aligned}\mathbb{I}_p(\mathcal{S}, \mathcal{S}'; \omega_p) = 1 - \max\bigl\{&\mathbb{1}\{\nu_p^{\ell}(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^{u}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p < a_p\},\\ &\mathbb{1}\{\nu_p^{u}(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^{\ell}(\cdot, \mathcal{S}; \omega_p)\}\mathbb{1}\{a'_p > a_p\}\bigr\}.\end{aligned}$$

Finally, we let

$$\mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega) = \min_{p=1,\dots,P}\{\mathbb{I}_p(\mathcal{S}, \mathcal{S}'; \omega_p)\}.$$

Remark 4. By construction, $\mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega) = 1$ if and only if the conditions in Proposition 2 are satisfied for each p, and it is zero otherwise. Therefore,

$$\begin{aligned}&\mathbb{1}\{\text{there exists a NE } \pi \text{ with support } \mathcal{S} \text{ and a NE } \pi' \text{ with support } \mathcal{S}'\}\\ &\quad \leq \mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega).\end{aligned} \tag{9}$$

Adding Assumption 3, we can finally analyze the event $\mathbf{y} \in \mathscr{E}(\omega)$ for any given $\mathbf{y} \in \mathcal{A}$.

4.2 *Incorporating Assumption* 3

Our ultimate focus is the event $\mathbf{y} \in \mathscr{E}(\omega)$ for a given $\mathbf{y}$. Since Assumption 3 maintains that $\mathbf{Y} \in \mathscr{E}(\omega)$, if we invoke (9), then the problem reduces to verifying whether $\mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega) = 1$

for *some pair* $(\mathcal{S}, \mathcal{S}')$ *such that* $\mathbf{Y} \in \mathcal{S}$ *and* $\mathbf{y} \in \mathcal{S}'$. For any pair of action profiles $\mathbf{y}, \mathbf{y}'$ in $\mathcal{A}$, let

$$\begin{aligned} &\mathbb{I}^{\mathscr{E}}(\mathbf{y}, \mathbf{y}'; \omega) = \max_{\mathcal{S}, \mathcal{S}'}\{\mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega)\} \\ &\quad \text{such that (i) } \mathcal{S} \text{ and } \mathcal{S}' \text{ satisfy (4); (ii) } \mathbf{y} \in \mathcal{S} \text{ and } \mathbf{y}' \in \mathcal{S}'. \end{aligned} \tag{10}$$

What we do in (10) is to search over all possible NE supports $\mathcal{S}$ and $\mathcal{S}'$ that include $\mathbf{y}$ and $\mathbf{y}'$, respectively, and verify whether the conditions in Proposition 2 are satisfied for *some* such $\mathcal{S}$ and $\mathcal{S}'$. If this is the case, $\mathbb{I}^{\mathscr{E}}(\mathbf{y}, \mathbf{y}'; \omega) = 1$; otherwise, it equals 0. From (9) and (10), we have

$$\mathbb{1}\{\mathbf{y}, \mathbf{y}' \in \mathscr{E}(\omega)\} \leq \mathbb{I}^{\mathscr{E}}(\mathbf{y}, \mathbf{y}'; \omega).$$

From (5), the total number of distinct pairs $(\mathcal{S}, \mathcal{S}')$ involved in the search is at most $(3^P - 2P)^2$. We are now ready to present the main result of this section.

PROPOSITION 3. *Suppose Assumptions 1–3 hold. Then, for any* $\mathbf{y}$ *in* $\mathcal{A}$, *w.p.*1 *we have*[5]

$$\mathbb{1}\{\mathbf{Y} = \mathbf{y}\} \leq \mathbb{1}\{\mathbf{y} \in \mathscr{E}(\omega)\} \leq \mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}; \omega). \tag{11}$$

Furthermore, our assumptions are compatible with the conditions needed to have either $\mathbb{1}\{\mathbf{y} \in \mathscr{E}(\omega)\} = \mathbb{1}\{\mathbf{Y} = \mathbf{y}\}$ *w.p.*1 *or* $\mathbb{1}\{\mathbf{y} \in \mathscr{E}(\omega)\} = \mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}; \omega)$ *w.p.*1.

PROOF. The lower bound follows directly from Assumption 3. The upper bound follows from (9) and the definition of $\mathbb{I}^{\mathscr{E}}$ in (10). Having $\mathbb{1}\{\mathbf{Y} = \mathbf{y}\} = \mathbb{1}\{\mathbf{y} \in \mathscr{E}(\omega)\}$ holds under (3), where each agent has a strictly dominant action w.p.1. The last piece of the statement comes from the final part of Proposition 2 (see also Remark 3). $\square$

Constructing an upper bound for $\Pr[\mathbf{y} \in \mathscr{E}(\omega)|X]$ using (11) directly is not feasible since $\mathbb{I}^{\mathscr{E}}$ cannot be observed. Constructive results follow from Assumption 4.

4.3 *Constructive results using Assumption* 4

Going back to Definition 2, constructing $\mathbb{I}_p(\mathcal{S}, \mathcal{S}'; \omega_p)$ and $\mathbb{I}(\mathcal{S}, \mathcal{S}'; \omega)$ requires observing the indicator functions $\mathbb{1}\{\nu_p^\ell(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^u(\cdot, \mathcal{S}; \omega_p)\}$ and $\mathbb{1}\{\nu_p^u(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^\ell(\cdot, \mathcal{S}; \omega_p)\}$ for each p. But this is not feasible because payoff functions are unknown to the researcher. It follows that $\mathbb{I}^{\mathscr{E}}(\mathbf{y}, \mathbf{y}'; \omega)$ is unknown and, therefore, it is not possible to use (11) directly to construct an upper bound for $\Pr[\mathbf{y} \in \mathscr{E}(\omega)|X]$. Assumption 4 enables us to replace the unobservable $\mathbb{I}^{\mathscr{E}}(\mathbf{y}, \mathbf{y}'; \omega)$ with a valid upper bound. Let

$$\mathbb{H}_p(\mathcal{S}, \mathcal{S}') = \mathbb{1}\{f_p(\mathbf{u}_{-p}) \geq f_p(\mathbf{v}_{-p})\ \forall \mathbf{u}_{-p} \in \mathcal{S}, \mathbf{v}_{-p} \in \mathcal{S}'\}.$$

[5] Note from (10) that $\mathbb{I}^{\mathscr{E}}(\mathbf{y}, \mathbf{y}'; \omega)$ is symmetric in $\mathbf{y}$ and $\mathbf{y}'$. Therefore, $\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}; \omega) = \mathbb{I}^{\mathscr{E}}(\mathbf{y}, \mathbf{Y}; \omega)$.

By Assumption 4, $\mathbb{H}_p(\mathcal{S},\mathcal{S}')=1$ almost surely implies that having $\mathbf{Y}_{-p}\in\mathcal{S}$ leaves p worse off than if $\mathbf{Y}_{-p}\in\mathcal{S}'$ for any action p can choose. Thus,

$$\begin{aligned}&\mathbb{H}_p(\mathcal{S},\mathcal{S}')\leq \mathbb{1}\{\nu_p^{\ell}(\cdot,\mathcal{S}';\omega_p)\geq \nu_p^{u}(\cdot,\mathcal{S};\omega_p)\}\quad\text{and}\\ &\mathbb{H}_p(\mathcal{S}',\mathcal{S})\leq \mathbb{1}\{\nu_p^{u}(\cdot,\mathcal{S}';\omega_p)\leq \nu_p^{\ell}(\cdot,\mathcal{S};\omega_p)\}\quad\text{w.p.1.}\end{aligned}\tag{12}$$

This yields a valid, observable upper bound for the unfeasible $\mathbb{I}^{\mathscr{E}}(\mathbf{y},\mathbf{y}';\omega)$.

Definition 3 ($\widetilde{\mathbb{I}}_{\mathbf{p}}(\mathcal{S},\mathcal{S}'),\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')$). Let $\mathcal{S}=\prod_{p=1}^{P}\{a_p,b_p\}$ and $\mathcal{S}'=\prod_{p=1}^{P}\{a'_p,b'_p\}$ be any two sets that satisfy the conditions in (4). For each p, let $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$ be the indicator function that results when we replace $\mathbb{1}\{\nu_p^{\ell}(\cdot,\mathcal{S}';\omega_p)\geq \nu_p^{u}(\cdot,\mathcal{S};\omega_p)\}$ with $\mathbb{H}_p(\mathcal{S},\mathcal{S}')$ and replace $\mathbb{1}\{\nu_p^{u}(\cdot,\mathcal{S}';\omega_p)\leq \nu_p^{\ell}(\cdot,\mathcal{S};\omega_p)\}$ with $\mathbb{H}_p(\mathcal{S}',\mathcal{S})$ everywhere in Definition 2, and let

$$\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')=\min_{p=1,\dots,P}\{\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')\}.$$

By (12),

$$\mathbb{I}_p(\mathcal{S},\mathcal{S}';\omega_p)\leq\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')\quad\forall p;\quad\text{therefore,}\quad \mathbb{I}(\mathcal{S},\mathcal{S}';\omega)\leq\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')\quad\text{w.p.1.}\tag{13}$$

For any pair of action profiles $\mathbf{y},\mathbf{y}'$ in $\mathcal{A}$, let

$$\begin{aligned}&\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{y},\mathbf{y}')=\max_{\mathcal{S},\mathcal{S}'}\{\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')\}\\ &\qquad\text{such that (i) }\mathcal{S}\text{ and }\mathcal{S}'\text{ satisfy (4); (ii) }\mathbf{y}\in\mathcal{S}\text{ and }\mathbf{y}'\in\mathcal{S}'.\end{aligned}\tag{14}$$

By (13), we have $\mathbb{I}^{\mathscr{E}}(\mathbf{y},\mathbf{y}';\omega)\leq\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{y},\mathbf{y}')$ w.p.1. Based on Assumption 4, the best observable *valid* bounds that can be derived from Proposition 3 are

$$\mathbb{1}\{\mathbf{Y}=\mathbf{y}\}\leq\mathbb{1}\{\mathbf{y}\in\mathscr{E}(\omega)\}\leq\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y},\mathbf{y}).\tag{15}$$

Remark 5. Equation (15) is the main constructive result in this paper and our probability bounds are derived from it.

Example 1 (Continued). We illustrate how to construct the indicator functions $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$, $\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')$, and $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{y},\mathbf{y}')$ in each of the two cases described in the example of Section 3.

Computing $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$.

(i) We have $f_p(\mathbf{y}_{-p})=(\sum_{q\in\mathscr{S}_p}y_q,-\sum_{q\in\mathscr{C}_p}y_q)$, where $\mathscr{S}_p$ and $\mathscr{C}_p$ denote the group of substitutes and complements of Y_p. For any pair of sets $\mathcal{S}=\prod_{p=1}^{P}\{a_p,b_p\}$ and $\mathcal{S}'=\prod_{p=1}^{P}\{a'_p,b'_p\}$ as described in (4), we then have

$$\begin{aligned}&\mathbb{H}_p(\mathcal{S},\mathcal{S}')=\mathbb{1}\left\{\sum_{q\in\mathscr{S}_p}a_q\geq\sum_{q\in\mathscr{S}_p}b'_q,\text{ and }\sum_{q\in\mathscr{C}_p}b_q\leq\sum_{q\in\mathscr{C}_p}a'_q\right\},\\ &\mathbb{H}_p(\mathcal{S}',\mathcal{S})=\mathbb{1}\left\{\sum_{q\in\mathscr{S}_p}a'_q\geq\sum_{q\in\mathscr{S}_p}b_q,\text{ and }\sum_{q\in\mathscr{C}_p}b'_q\leq\sum_{q\in\mathscr{C}_p}a_q\right\}.\end{aligned}$$

Using these expressions, $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$ is obtained as we describe in Definition 3.

(ii) We have $f_p(\mathbf{y}_{-p}) = ((y_q)_{q\in\mathscr{S}_p}, (-y_q)_{q\in\mathscr{C}_p})$. For any pair of sets $\mathcal{S} = \prod_{p=1}^P\{a_p, b_p\}$ and $\mathcal{S}' = \prod_{p=1}^P\{a'_p, b'_p\}$ as described in (4), we then have

$$\mathbb{H}_p(\mathcal{S},\mathcal{S}') = \mathbb{1}\{a_q \geq b'_q\ \forall q\in\mathscr{S}_p,\ \text{and}\ b_q \leq a'_q\ \forall q\in\mathscr{C}_p\},$$
$$\mathbb{H}_p(\mathcal{S}',\mathcal{S}) = \mathbb{1}\{a'_q \geq b_q\ \forall q\in\mathscr{S}_p,\ \text{and}\ b'_q \leq a_q\ \forall q\in\mathscr{C}_p\}.$$

Using these expressions, $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$ is obtained as we describe in Definition 3.

Computing $\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')$ *and* $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{y},\mathbf{y}')$.

Once $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$ is obtained for each p, we have $\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}') = \min_{p=1,\dots,P}\{\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')\}$. For any $\mathbf{y}$, $\mathbf{y}'$ in $\mathcal{A}$, from (14) we have

$$\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{y},\mathbf{y}') = \max_{\mathcal{S},\mathcal{S}'}\{\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')\}$$
$$\text{such that (i) } \mathcal{S} \text{ and } \mathcal{S}' \text{ satisfy (4); (ii) } \mathbf{y}\in\mathcal{S} \text{ and } \mathbf{y}'\in\mathcal{S}'.$$

From (5), if $\mathbf{y}$, $\mathbf{y}'$ belong in the interior of $\mathcal{A}$, constructing $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{y},\mathbf{y}')$ involves computing $\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')$ for $(3^P - 2P)^2$ distinct pairs $(\mathcal{S},\mathcal{S}')$. If $P = 2$, this number is 25; if $P = 4$, it grows to 5329. Although this number grows exponentially with P, we stress that even in applications with large P, the simple structure of each pair $(\mathcal{S},\mathcal{S}')$ (described in (4)), as well as the straightforward expressions for each $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$, makes the task of coding and computing $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{y},\mathbf{y}')$ entirely feasible. As we see in Section 6, restricting attention to pure-strategy NE eliminates any computational concern altogether.

5. Probability bounds for Nash equilibrium action profiles

Let $\mathbf{y}$ and $\mathcal{C}$ denote a prespecified action profile and a collection of profiles, respectively. As before, X denotes the collection of observable payoff-relevant covariates. Using the results from Section 4, we characterize bounds for the probabilities[6]

$$P_{\mathscr{E}}(\mathbf{y}, X) \equiv \Pr[\mathbf{y}\in\mathscr{E}(\omega)|X] = \Pr[\,\mathbf{y} \text{ is a NE profile}|X]$$

and

$$P_{\mathscr{E}}(\mathcal{C}, X) \equiv \Pr[\mathcal{C}\cap\mathscr{E}(\omega)\neq\emptyset|X] = \Pr[\,\mathcal{C} \text{ includes a NE profile}|X].$$

Comparing the upper bound with $\Pr[\mathbf{Y}=\mathbf{Y}|X]$, we also derive a lower bound for

$$Q_{\mathscr{E}}(\mathbf{y}, X) \equiv \Pr[\mathcal{M}_{\mathscr{E}} \text{ selects a NE } \pi : \pi(\mathbf{y}) > 0|\mathbf{y}\in\mathscr{E}(\omega), X],$$
$$Q_{\mathscr{E}}(\mathcal{C}, X) \equiv \Pr[\mathcal{M}_{\mathscr{E}} \text{ selects a NE } \pi : \pi(\mathbf{y}) > 0 \text{ for some } \mathbf{y}\in\mathcal{C}|\mathcal{C}\cap\mathscr{E}(\omega)\neq\emptyset, X];$$

that is, the likelihood that $\mathcal{M}_{\mathscr{E}}$ selects a NE where $\mathbf{y}$ (or some profile in $\mathcal{C}$) is played *given that such a NE exists.*

[6]Our results could, for example, also enable us to determine bounds for $\Pr[\mathcal{C}\subseteq\mathscr{E}(\omega)|X]$, the probability that *every* element in $\mathcal{C}$ is a NE profile. For brevity, we focus only on $\Pr[\mathbf{y}\in\mathscr{E}(\omega)|X]$ and $\Pr[\mathcal{C}\cap\mathscr{E}(\omega)\neq\emptyset|X]$.

5.1 *Bounds implied by Proposition* 3

We begin with bounds for $P_{\mathscr{E}}(\mathbf{y}, X)$ and $Q_{\mathscr{E}}(\mathbf{y}, X)$. Proposition 3 yields the following result.

Proposition 4. *Take any profile* $\mathbf{y}$ *in* $\mathcal{A}$. *As before, let* X *denote the vector of observable payoff shifters in* ω.

(i) *If Assumptions 1–3 are satisfied, then*

$$\Pr[\mathbf{Y} = \mathbf{y}|X] \leq P_{\mathscr{E}}(\mathbf{y}, X) \leq E[\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}; \omega)|X].$$

Furthermore, our assumptions are compatible with the conditions needed for either of these bounds to be attained.

(ii) *For any profile* $\mathbf{y}$ *such that* $E[\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}; \omega)|X] > 0$,

$$\frac{\Pr[\mathbf{Y} = \mathbf{y}|X]}{E[\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}; \omega)|X]} \leq Q_{\mathscr{E}}(\mathbf{y}, X) \leq 1.$$

Furthermore, our assumptions are compatible with the conditions needed for either of these bounds to be attained.

Part (i) of Proposition 4 follows directly from Proposition 3. The upper bound of 1 in part (ii) is attained, for example, if each agent has a dominant action with probability 1 (see Remark 3 and Equation (3)). In such a case, having $\mathbf{y} \in \mathscr{E}(\omega)$ automatically implies that there are no other profiles in $\mathscr{E}(\omega)$ and trivially the mechanism $\mathcal{M}_{\mathscr{E}}$ must select $\mathbf{y}$. To understand the lower bound in (ii), suppose there exists a NE π that includes $\mathbf{y}$ in its support. In that case, our assumptions place no restrictions on the range of values $\pi(\mathbf{y})$ (the probability with which $\mathbf{y}$ is played in π) can take on $(0, 1)$. Thus, our assumptions only imply that

$$\begin{aligned}\Pr[\mathbf{Y} = \mathbf{y}|\mathbf{y} \in \mathscr{E}(\omega), X] &\leq \Pr[\mathcal{M}_{\mathscr{E}} \text{ selects a NE } \pi^* : \pi^*(\mathbf{y}) > 0|\mathbf{y} \in \mathscr{E}(\omega), X] \\ &\equiv Q_{\mathscr{E}}(\mathbf{y}, X).\end{aligned}$$

Since

$$\Pr[\mathbf{Y} = \mathbf{y}|\mathbf{y} \in \mathscr{E}(\omega), X] = \frac{\Pr[\mathbf{Y} = \mathbf{y}|X]}{\Pr[\mathbf{y} \in \mathscr{E}(\omega)|X]},$$

the lower bound in part (ii) follows from the upper bound in part (i). Let $\mathcal{C} \equiv (\mathbf{y}^1, \ldots, \mathbf{y}^L)$ be a prespecified collection in $\mathcal{A}$. By Assumption 3 and Proposition 3,

$$\mathbb{1}\{\mathbf{Y} \in \mathcal{C}\} \leq \mathbb{1}\{\mathcal{C} \cap \mathscr{E}(\omega) \neq \emptyset\} \leq \max_{\mathbf{y}^\ell \in \mathcal{C}}\{\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^\ell; \omega)\} \quad \text{w.p.1.} \tag{16}$$

From this, we have the following result.

PROPOSITION 5. (i) *If Assumptions 1–3 are satisfied, then*

$$\Pr[\mathbf{Y} \in \mathcal{C}|X] \le P_{\mathscr{E}}(\mathcal{C}, X) \le E\Big[\max_{\mathbf{y}^\ell \in \mathcal{C}}\{\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^\ell; \omega)\}\big|X\Big].$$

Furthermore, our assumptions are compatible with the conditions needed for either of these bounds to be attained.

(ii) *For any* $\mathcal{C}$ *such that* $E[\max_{\mathbf{y}^\ell \in \mathcal{C}}\{\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^\ell; \omega)\}|X] > 0$,

$$\frac{\Pr[\mathbf{Y} \in \mathcal{C}|X]}{E[\max_{\mathbf{y}^\ell \in \mathcal{C}}\{\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^\ell; \omega)\}|X]} \le Q_{\mathscr{E}}(\mathcal{C}, X) \le 1.$$

As before, our assumptions are compatible with the conditions needed for either of these bounds to be attained.

Part (i) follows directly from (16). Part (ii) is shown using the same arguments as in part (ii) of Proposition 4. All bounds involving $\mathbb{I}^{\mathscr{E}}$ are unfeasible because this indicator function is not observed. We obtain valid, observable bounds by replacing $\mathbb{I}^{\mathscr{E}}$ with $\widetilde{\mathbb{I}}^{\mathscr{E}}$.

5.2 *Feasible probability bounds using Assumption 4*

The bounds in Section 5.1 which involve $\mathbb{I}^{\mathscr{E}}$ cannot be constructed or estimated because this indicator function is not observed. Based on (15), we can replace $\mathbb{I}^{\mathscr{E}}$ with $\widetilde{\mathbb{I}}^{\mathscr{E}}$ in every instance and the resulting bounds remains valid. For a given $\mathbf{y}$, let

$$P^\ell_{\mathscr{E}}(\mathbf{y}, X) \equiv \Pr[\mathbf{Y} = \mathbf{y}|X] \quad \text{and} \quad P^u_{\mathscr{E}}(\mathbf{y}, X) \equiv E[\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y})|X].$$

Using part (i) of Proposition 4, our bounds for $P_{\mathscr{E}}(\mathbf{y}, X)$ are

$$P^\ell_{\mathscr{E}}(\mathbf{y}, X) \le P_{\mathscr{E}}(\mathbf{y}, X) \le P^u_{\mathscr{E}}(\mathbf{y}, X). \tag{17.i}$$

If $P^u_{\mathscr{E}}(\mathbf{y}, X) > 0$, let

$$Q^\ell_{\mathscr{E}}(\mathbf{y}, X) \equiv \frac{P^\ell_{\mathscr{E}}(\mathbf{y}, X)}{P^u_{\mathscr{E}}(\mathbf{y}, X)}.$$

From part (ii) of Proposition 4, our bounds for $Q_{\mathscr{E}}(\mathbf{y}, X)$ are

$$Q^\ell_{\mathscr{E}}(\mathbf{y}, X) \le Q_{\mathscr{E}}(\mathbf{y}, X) \le 1. \tag{17.ii}$$

Similarly, for a class $\mathcal{C}$ of profiles in $\mathcal{A}$, let

$$P^\ell_{\mathscr{E}}(\mathcal{C}, X) \equiv \Pr[\mathbf{Y} \in \mathcal{C}|X] \quad \text{and} \quad P^u_{\mathscr{E}}(\mathcal{C}, X) \equiv E\Big[\max_{\mathbf{y}^\ell \in \mathcal{C}}\{\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^\ell)\}\big|X\Big].$$

Using part (i) of Proposition 5, our bounds for $P_{\mathscr{E}}(\mathcal{C}, X)$ are

$$P^\ell_{\mathscr{E}}(\mathcal{C}, X) \le P_{\mathscr{E}}(\mathcal{C}, X) \le P^u_{\mathscr{E}}(\mathcal{C}, X), \tag{18.i}$$

and if $P^u_{\mathscr{E}}(\mathcal{C},X)>0$, let

$$Q^\ell_{\mathscr{E}}(\mathcal{C},X)\equiv\frac{P^\ell_{\mathscr{E}}(\mathcal{C},X)}{P^u_{\mathscr{E}}(\mathcal{C},X)}.$$

Using part (ii) of Proposition 5, the bounds for $Q_{\mathscr{E}}(\mathcal{C},X)$ are

$$Q^\ell_{\mathscr{E}}(\mathcal{C},X)\le Q_{\mathscr{E}}(\mathcal{C},X)\le 1. \tag{18.ii}$$

Under our assumptions, (17.i)–(18.ii) constitute the best *valid* observable bounds that can be derived from Propositions 4 and 5.

6. The case of pure-strategy Nash equilibrium behavior

Our results simplify considerably if we assume that the outcome observed is a pure-strategy NE. Let

$$\mathscr{E}^*(\omega)=\{\mathbf{y}\in\mathcal{A}:\mathbf{y}\text{ is a pure-strategy NE profile}\}.$$

By construction, the only possible support for a pure-strategy NE where $\mathbf{y}$ is played is $\mathcal{S}=\{\mathbf{y}\}$. If we let $\mathcal{S}=\{\mathbf{y}\}$ and $\mathcal{S}'=\{\mathbf{y}'\}$, then from Definition 3 (Case IV), we have

$$\begin{aligned}\widetilde{\mathbb{I}}_p(\{\mathbf{y}\},\{\mathbf{y}'\})=1&-\mathbb{1}\{f_p(\mathbf{y}'_{-p})\le f_p(\mathbf{y}_{-p})\}\cdot\mathbb{1}\{y'_p<y_p\}\\&-\mathbb{1}\{f_p(\mathbf{y}'_{-p})\ge f_p(\mathbf{y}_{-p})\}\cdot\mathbb{1}\{y'_p>y_p\}.\end{aligned} \tag{19}$$

Let us denote

$$\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{y},\mathbf{y}')\equiv\min_{p=1,\ldots,P}\bigl\{\widetilde{\mathbb{I}}_p(\{\mathbf{y}\},\{\mathbf{y}'\})\bigr\}. \tag{20}$$

From our results in previous sections, we have $\mathbb{1}\{\mathbf{y},\mathbf{y}'\in\mathscr{E}^*(\omega)\}\le\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{y},\mathbf{y}')$. Consider the following stronger version of Assumption 3.

Assumption 3′. *With probability* 1, $\mathbf{Y}\in\mathscr{E}^*(\omega)$; *that is, the selection mechanism* $\mathcal{M}_{\mathscr{E}^*}$ *now directly selects an outcome* (*as opposed to a mixing distribution*), *and this outcome is a pure-strategy NE with probability* 1.

If Assumption 3′ holds, we have[7]

$$\mathbb{1}\{\mathbf{Y}=\mathbf{y}\}\le\mathbb{1}\{\mathbf{y}\in\mathscr{E}^*(\omega)\}\le\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y},\mathbf{y}). \tag{21}$$

For any such profile or any collection $\mathcal{C}$ in $\mathcal{A}$, denote

$$P_{\mathscr{E}^*}(\mathbf{y},X)\equiv\Pr[\mathbf{y}\in\mathscr{E}^*(\omega)|X],$$

[7]This result remains valid even without strictly concave payoffs as long as weak concavity and local monotonicity still hold. See footnote 2 and the lower left panel in Figure 1 for an illustration of weakly concave payoffs.

$$P_{\mathscr{E}^*}(\mathcal{C}, X) \equiv \Pr[\mathcal{C} \cap \mathscr{E}^*(\omega) \neq \emptyset | X],$$

$$Q_{\mathscr{E}^*}(\mathbf{y}, X) \equiv \Pr[\mathcal{M}_{\mathscr{E}^*} \text{ selects } \mathbf{y} | \mathbf{y} \in \mathscr{E}^*(\omega), X],$$

$$Q_{\mathscr{E}^*}(\mathcal{C}, X) \equiv \Pr[\mathcal{M}_{\mathscr{E}^*} \text{ selects some } \mathbf{y} \in \mathcal{C} | \mathcal{C} \cap \mathscr{E}^*(\omega) \neq \emptyset, X].$$

These are the probabilities from Section 5 for the case of pure-strategy NE behavior. Based on (21), the expressions for the bounds in (17.i)–(18.ii) remain valid for these probabilities if we replace $\widetilde{\mathbb{I}}^{\mathscr{E}}$ with $\widetilde{\mathbb{I}}^{\mathscr{E}^*}$ in each case. As expected, assuming pure-strategy NE refines these bounds; this is illustrated in the empirical example of Section 8.

Example 1 (Continued). We show how to construct $\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{y}, \mathbf{y}')$ in each of the two cases described in the example of Section 3.

(i) We have $f_p(\mathbf{y}_{-p}) = (\sum_{q \in \mathscr{S}_p} y_q, -\sum_{q \in \mathscr{C}_p} y_q)$, where $\mathscr{S}_p$ and $\mathscr{C}_p$ denote the group of substitutes and complements of Y_p. For any pair $\mathbf{y} \equiv (y_p)_{p=1}^P$ and $\mathbf{y}' \equiv (y'_p)_{p=1}^P$ in $\mathcal{A}$, Equation (19) yields

$$\begin{aligned}\widetilde{\mathbb{I}}_p(\{\mathbf{y}\}, \{\mathbf{y}'\}) = 1 &- \mathbb{1}\left\{\sum_{q \in \mathscr{S}_p} y'_q \leq \sum_{q \in \mathscr{S}_p} y_q \text{ and } \sum_{q \in \mathscr{C}_p} y'_q \geq \sum_{q \in \mathscr{C}_p} y_q\right\} \cdot \mathbb{1}\{y'_p < y_p\} \\ &- \mathbb{1}\left\{\sum_{q \in \mathscr{S}_p} y'_q \geq \sum_{q \in \mathscr{S}_p} y_q \text{ and } \sum_{q \in \mathscr{C}_p} y'_q \leq \sum_{q \in \mathscr{C}_p} y_q\right\} \cdot \mathbb{1}\{y'_p > y_p\},\end{aligned}$$

(ii) We have $f_p(\mathbf{y}_{-p}) = ((y_q)_{q \in \mathscr{S}_p}, (-y_q)_{q \in \mathscr{C}_p})$. For any pair $\mathbf{y} \equiv (y_p)_{p=1}^P$ and $\mathbf{y}' \equiv (y'_p)_{p=1}^P$ in $\mathcal{A}$, Equation (19) yields

$$\begin{aligned}\widetilde{\mathbb{I}}_p(\{\mathbf{y}\}, \{\mathbf{y}'\}) = 1 &- \mathbb{1}\{y'_q \leq y_q \ \forall q \in \mathscr{S}_p \text{ and } y'_q \geq y_q \ \forall q \in \mathscr{C}_p\} \cdot \mathbb{1}\{y'_p < y_p\} \\ &- \mathbb{1}\{y'_q \geq y_q \ \forall q \in \mathscr{S}_p \text{ and } y'_q \leq y_q \ \forall q \in \mathscr{C}_p\} \cdot \mathbb{1}\{y'_p > y_p\}.\end{aligned}$$

In all cases, from (20), we have $\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{y}, \mathbf{y}') = \min_{p=1,\ldots,P}\{\widetilde{\mathbb{I}}_p(\{\mathbf{y}\}, \{\mathbf{y}'\})\}$.

Remark 6. Put together, our results show that in a game with the features described here, the data can be informative about *some* features of the underlying selection mechanism. However, without additional assumptions, there are some aspects of the selection mechanism that we cannot uncover. For illustration, focus on a 2×2 game of strategic substitutes and suppose we maintain that the outcomes observed are always pure-strategy NE. In this setting, $(1, 0)$ and $(0, 1)$ could simultaneously be NE. Let us focus on the outcome $(1, 0)$. While our results produce a lower bound on $\Pr[\mathcal{M}_{\mathscr{E}^*} \text{ selects } (1, 0) | (1, 0) \in \mathscr{E}^*(\omega)]$, finding bounds for $\Pr[\mathcal{M}_{\mathscr{E}^*} \text{ selects } (1, 0) | (1, 0), (0, 1) \in \mathscr{E}^*(\omega)]$ is not a well defined problem. This is because, while our assumptions enable us to produce an upper bound for $\Pr[(1, 0), (0, 1) \in \mathscr{E}^*(\omega)]$, the lower bound for this probability is always zero. Naturally, if we are willing to impose the restriction that the latter probability is bounded away from zero, bounds for $\Pr[\mathcal{M}_{\mathscr{E}^*} \text{ selects } (1, 0) | (1, 0), (0, 1) \in \mathscr{E}^*(\omega)]$ would follow from our results. More generally, our bounds provide clear guidelines for any simulation exercise of a game fitting our general assumptions. For instance, suppose the data reveal that $\Pr[\mathcal{M}_{\mathscr{E}^*} \text{ selects } (1,0)|(1,0) \in \mathscr{E}^*(\omega)] = 1$. Then, in any simulation aimed at describing said data, the selection mechanism must always choose $(1,0)$ whenever it is a NE, including the case where $(0,1)$ is also a NE. If the original data also yielded $\Pr[\mathcal{M}_{\mathscr{E}^*} \text{ selects } (0,1)|(0,1) \in \mathscr{E}^*(\omega)] = 1$, then any simulation must be such that $\Pr[(1,0),(0,1) \in \mathscr{E}^*(\omega)] = 0$.

7. Inference

Let x denote a particular value of X and let $\mathcal{C}$ denote a collection of profiles in $\mathcal{A}$. Based on the bounds in (18.i) and (18.ii), we construct confidence intervals for $P_{\mathscr{E}}(\mathcal{C},x)$ and $Q_{\mathscr{E}^*}(\mathcal{C},x)$ that asymptotically include the true values of these probabilities with a fixed probability $1-\alpha$. If pure-strategy NE behavior is maintained as in Section 6, confidence intervals for $P_{\mathscr{E}^*}(\mathcal{C},x)$ and $Q_{\mathscr{E}^*}(\mathcal{C},x)$ are constructed in the exact same way after replacing $\widetilde{\mathbb{I}}^{\mathscr{E}}$ with $\widetilde{\mathbb{I}}^{\mathscr{E}^*}$. We maintain that we observe an iid[8] sample $(\mathbf{Y}_i, X_i)_{i=1}^N$ which satisfies Assumptions 1–4. We apply the results in Imbens and Manski (2004) and Stoye (2009) to our setting.[9]. Split $X=(X^c,X^d)$, where $X^c \in \mathbb{R}^c$ and $X^d \in \mathbb{R}^d$ denote the continuous and discrete elements in X, respectively. We partition $x=(x^c,x^d)$ accordingly. Let $K:\mathbb{R}^c \longrightarrow \mathbb{R}$ and h_N denote a kernel function and a bandwidth sequence, respectively. Let

$$\widehat{g}(x) = \frac{1}{Nh_N^c}\sum_{i=1}^N K\left(\frac{X_i^c - x^c}{h_N}\right)\cdot \mathbb{1}\{X_i^d = x^d\}.$$

Our results come from the inequalities $P^\ell_{\mathscr{E}}(\mathcal{C},x) \le P_{\mathscr{E}}(\mathcal{C},x) \le P^u_{\mathscr{E}}(\mathcal{C},x)$ and $Q^\ell_{\mathscr{E}}(\mathcal{C},x) \le Q_{\mathscr{E}}(\mathcal{C},x) \le 1$ in (18.i) and (18.ii). Our estimators for $P^\ell_{\mathscr{E}}(\mathcal{C},x)$, $P^u_{\mathscr{E}}(\mathcal{C},x)$, and $Q^\ell_{\mathscr{E}}(\mathcal{C},x)$ are of the form

$$\widehat{P}^\ell_{\mathscr{E}}(\mathcal{C},x) = \frac{1}{Nh_N^c}\sum_{i=1}^N \mathbb{1}\{\mathbf{Y}_i \in \mathcal{C}\}K\left(\frac{X_i^c - x^c}{h_N}\right)\mathbb{1}\{X_i^d = x^d\}/\widehat{g}(x),$$

$$\widehat{P}^u_{\mathscr{E}}(\mathcal{C},x) = \frac{1}{Nh_N^c}\sum_{i=1}^N \max_{\mathbf{y}^j\in\mathcal{C}}\{\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y}_i,\mathbf{y}^j)\}K\left(\frac{X_i^c - x^c}{h_N}\right)\mathbb{1}\{X_i^d = x^d\}/\widehat{g}(x),$$

$$\widehat{Q}^\ell_{\mathscr{E}}(\mathcal{C},x) = \widehat{P}_\ell(\mathcal{C},x)/\widehat{P}_u(\mathcal{C},x).$$

Assumption 5. *Let $f_{c|d}$ denote the conditional density of X^c given X^d. For a given $x \equiv (x^d,x^c)$ in the support of X, denote $f_{c|d}(x^c|x^d)\cdot\Pr(X^d = x^d) \equiv g(x)$. The data observed $(\mathbf{Y}_i,X_i)_{i=1}^N$ is an iid sample from a distribution belonging to a family $\mathcal{G}$, where*

[8]Asymptotically valid inference could potentially be performed if $(\mathbf{Y}_i, X_i)_{i=1}^N$ are identically distributed but not independent. We focus on the iid case for simplicity.

[9]We concentrate on confidence intervals for individual probabilities. More generally, we could also construct asymptotically valid *joint* confidence regions, for example, for $P_{\mathscr{E}}(\mathcal{C},x)$ and $Q_{\mathscr{E}}(\mathcal{C},x)$ or for $P_{\mathscr{E}}(\mathcal{C},x)$ and $P_{\mathscr{E}}(\mathcal{C},x')$ with $x \neq x'$.

each one of its members satisfies the following conditions for $\mathcal{C}$ and almost every x in some compact set $\mathcal{X}$: (i) $0 < \underline{g} \leq g(x) \leq \overline{g} < \infty$, $0 < \underline{p} \leq P^u_{\mathscr{E}}(\mathcal{C}, x) \leq \overline{p} < 1$, *and* $0 < \underline{p} \leq P^\ell_{\mathscr{E}}(\mathcal{C}, x) \leq \overline{p} < 1$. (ii) $g(x)$, $P^\ell_{\mathscr{E}}(\mathcal{C}, x)$ *and* $P^u_{\mathscr{E}}(\mathcal{C}, x)$ *are twice differentiable with respect to* x^c (*the continuous elements in* x) *with bounded derivatives everywhere in* $\mathcal{X}$. *The kernel* K *is nonnegative, has compact support, is Lipschitz-continuous, bounded, and symmetric around zero. Denote* $\psi = (\psi_1, \dots, \psi_c)'$. *Then* $\int K(\psi)\, d\psi = 1$, $\int \psi K(\psi)\, d\psi = 0$, *and* $\int \|\psi\|^2 \psi K(\psi)\, d\psi < \infty$. *The bandwidth sequence* h_N *satisfies* $h_N \longrightarrow 0$, $N h^c_N / \log(N) \longrightarrow \infty$, *and* $N \log(N) h^{c+4}_N \longrightarrow 0$.

Let $\mu^K_2 \equiv \int K^2(\psi)\, d\psi$. If Assumption 5 holds, we can show that uniformly over the $\mathcal{G}$ and $\mathcal{X}$,[10]

$$
\begin{aligned}
&\sqrt{N h^c_N} \begin{pmatrix} \widehat{P}^\ell_{\mathscr{E}}(\mathcal{C}, x) - P^\ell_{\mathscr{E}}(\mathcal{C}, x) \\ \widehat{P}^u_{\mathscr{E}}(\mathcal{C}, x) - P^u_{\mathscr{E}}(\mathcal{C}, x) \end{pmatrix} \\
&\quad \xrightarrow{d} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left[\begin{pmatrix} \sigma^2_\ell(\mathcal{C}, x) & \rho(\mathcal{C}, x)\sigma_\ell(\mathcal{C}, x)\sigma_u(\mathcal{C}, x) \\ \rho(\mathcal{C}, x)\sigma_\ell(\mathcal{C}, x)\sigma_u(\mathcal{C}, x) & \sigma^2_u(\mathcal{C}, x) \end{pmatrix} \right] \right), \\
&\sqrt{N h^c_N} (\widehat{Q}^\ell_{\mathscr{E}}(\mathcal{C}, x) - Q^\ell_{\mathscr{E}}(\mathcal{C}, x)) \xrightarrow{d} \mathcal{N}(0, \Omega^2(\mathcal{C}, x)),
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma^2_\ell(\mathcal{C}, x) &= \frac{P^\ell_{\mathscr{E}}(\mathcal{C}, x)(1 - P^\ell_{\mathscr{E}}(\mathcal{C}, x))}{g(x)} \cdot \mu^K_2, \\
\sigma^2_u(\mathcal{C}, x) &= \frac{P^u_{\mathscr{E}}(\mathcal{C}, x)(1 - P^u_{\mathscr{E}}(\mathcal{C}, x))}{g(x)} \cdot \mu^K_2, \\
\rho(\mathcal{C}, x) &= \sqrt{\frac{P^\ell_{\mathscr{E}}(\mathcal{C}, x)(1 - P^u_{\mathscr{E}}(\mathcal{C}, x))}{P^u_{\mathscr{E}}(\mathcal{C}, x)(1 - P^\ell_{\mathscr{E}}(\mathcal{C}, x))}}, \\
\Omega^2(\mathcal{C}, x) &= \frac{Q^\ell_{\mathscr{E}}(\mathcal{C}, x)(1 - Q^\ell_{\mathscr{E}}(\mathcal{C}, x))}{P^u_{\mathscr{E}}(\mathcal{C}, x)} \cdot g(x) \cdot \mu^K_2.
\end{aligned}
$$

Since $K \geq 0$, we have $\widehat{P}_\ell(\mathcal{C}, x) \leq \widehat{P}_u(\mathcal{C}, x)$ w.p.1 for any x. Therefore, $0 \leq \widehat{Q}^\ell_{\mathscr{E}}(\mathcal{C}, x) \leq 1$. Based on (18.ii) (where the upper bound is the known constant 1), the following interval has uniformly valid asymptotic coverage of at least $(1 - \alpha)\%$ for $Q_{\mathscr{E}}(\mathcal{C}, x)$:

$$
\mathrm{CI}^Q_\alpha \equiv \left[\widehat{Q}^\ell_{\mathscr{E}}(\mathcal{C}, x) - \frac{\widehat{\Omega}(\mathcal{C}, x)}{\sqrt{N h^c_N}} \cdot z_\alpha, 1 \right], \quad \text{where } \Phi(z_\alpha) = 1 - \alpha. \tag{22}
$$

Moving on to $P_{\mathscr{E}}(\mathcal{C}, x)$, denote $\Delta(\mathcal{C}, x) = P_u(\mathcal{C}, x) - P_\ell(\mathcal{C}, x)$ and $\widehat{\Delta}(\mathcal{C}, x) = \widehat{P}_u(\mathcal{C}, x) - \widehat{P}_\ell(\mathcal{C}, x)$. As we pointed out above, we have $\widehat{P}_\ell(\mathcal{C}, x) \leq \widehat{P}_u(\mathcal{C}, x)$ w.p.1 for any x. Combining this with our previous asymptotic normality result, we can show that a feature analogous to Assumption 3 in Stoye (2009) holds in our setting. Namely, we can show that there ex-

[10] To understand the expression for $\rho(\mathcal{C}, x)$, note first that by construction $\mathbb{1}\{\mathbf{Y} \in \mathcal{C}\} \cdot \max_{\mathbf{y}^j \in \mathcal{C}} \{\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^j)\} = \mathbb{1}\{\mathbf{Y} \in \mathcal{C}\}$ and, therefore, $E[\mathbb{1}\{\mathbf{Y} \in \mathcal{C}\} \cdot \max_{\mathbf{y}^j \in \mathcal{C}} \{\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^j)\} | X = x] = E[\mathbb{1}\{\mathbf{Y} \in \mathcal{C}\} | X = x] = P_\ell(\mathcal{C}, x)$. From here, it follows that $\mathrm{Cov}(\mathbb{1}\{\mathbf{Y} \in \mathcal{C}\}, \max_{\mathbf{y}^j \in \mathcal{C}} \{\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}^j)\}) = P_\ell(\mathcal{C}, x) \cdot [1 - P_u(\mathcal{C}, x)]$.

ists a sequence $a_N \to 0$ such that $a_N\sqrt{Nh_N^c} \to \infty$ and $\sqrt{Nh_N^c}|\widehat{\Delta}(\mathcal{C},x) - \Delta_N(\mathcal{C},x)| \xrightarrow{p} 0$ for all sequences of distributions in $\mathcal{G}$ for which $\Delta_N(\mathcal{C},x) \le a_N$. This can be shown by following steps analogous to Lemma 3 in Stoye (2009). Combining these facts, we have that under Assumption 5, the following confidence interval has uniformly valid asymptotic coverage of $(1-\alpha)\%$ for the true value of $P(\mathcal{C},x)$:

$$\mathrm{CI}_\alpha^1 \equiv \left[\widehat{P}_{\mathscr{E}}^\ell(\mathcal{C},x) - \frac{\widehat{\sigma}_\ell(\mathcal{C},x)}{\sqrt{Nh_N^c}} \cdot c_\alpha^1(\mathcal{C},x), \widehat{P}_{\mathscr{E}}^u(\mathcal{C},x) + \frac{\widehat{\sigma}_u(\mathcal{C},x)}{\sqrt{Nh_N^c}} \cdot c_\alpha^1(\mathcal{C},x)\right], \tag{23}$$

where $c_\alpha^1(\mathcal{C},x)$ solves (for c_α) $\Phi(c_\alpha + \frac{\sqrt{Nh_N^c}\widehat{\Delta}(\mathcal{C},x)}{\max\{\widehat{\sigma}_\ell(\mathcal{C},x),\widehat{\sigma}_u(\mathcal{C},x)\}}) - \Phi(-c_\alpha) = 1-\alpha$. The proof is analogous to the proof of Proposition 1 in Stoye (2009).[11] If we allow for $P_{\mathscr{E}}^\ell(\mathcal{C},x) = 0$ but maintain the remaining restrictions in Assumption 5, the following equality is a valid confidence interval for $P_{\mathscr{E}}(\mathcal{C},x)$ with asymptotic coverage at least $(1-\alpha)\%$:

$$\mathrm{CI}_\alpha^3 \equiv \left[0, \widehat{P}_{\mathscr{E}}^u(\mathcal{C},x) + \frac{\widehat{\sigma}_u(\mathcal{C},x)}{\sqrt{Nh_N^c}} \cdot z_\alpha\right], \quad \text{where } \Phi(z_\alpha) = 1-\alpha. \tag{24}$$

On the other hand, the identified interval for $Q_{\mathscr{E}}(\mathcal{C},x)$ simply becomes $[0,1]$.

8. Empirical illustration: A model of multiple entry

We use our model to study the decision of how many stores to open in a market by the two dominant[12] firms in the U.S. home improvement products industry: Home Depot (player $p=1$) and Lowe's (player $p=2$). We model this problem as a simultaneous discrete game assumed to satisfy our assumptions. Accordingly, $Y_p =$ denotes the number of stores opened by player (firm) p, and no upper bound was imposed on the space of actions (i.e., M_p was assumed to be arbitrarily large relative to the collection of outcomes observed). Payoff functions are unknown, but we maintain that they satisfy Assumptions 1 and 2. We also maintain that the outcomes observed result from NE behavior (Assumption 3) and in instances that are made explicitly clear, we assume they result from

[11] Alternatively, following the steps in the proof of Proposition 2 in Stoye (2009), we can show that the following confidence interval is also valid

$$\mathrm{CI}_\alpha^2 \equiv \left[\widehat{P}_{\mathscr{E}}^\ell(\mathcal{C},x) - \frac{\widehat{\sigma}_\ell(\mathcal{C},x)}{\sqrt{Nh_N^c}} \cdot c_\alpha^\ell(\mathcal{C},x), \widehat{P}_{\mathscr{E}}^u(\mathcal{C},x) + \frac{\widehat{\sigma}_u(\mathcal{C},x)}{\sqrt{Nh_N^c}} \cdot c_\alpha^u(\mathcal{C},x)\right],$$

where $c_\alpha^\ell(\mathcal{C},x)$ and $c_\alpha^u(\mathcal{C},x)$ minimize $\widehat{\sigma}_\ell(\mathcal{C},x)\cdot c_\alpha^\ell + \widehat{\sigma}_u(\mathcal{C},x)\cdot c_\alpha^u$ subject to

$$\Pr\left(-c_\alpha^\ell \le Z_1, \widehat{\rho}(\mathcal{C},x)\cdot Z_1 \le c_\alpha^u + \frac{\sqrt{Nh_N^c}\widehat{\Delta}(\mathcal{C},x)}{\widehat{\sigma}_u(\mathcal{C},x)} + \sqrt{1-\widehat{\rho}^2(\mathcal{C},x)}\cdot Z_2\right) \ge 1-\alpha,$$

$$\Pr\left(-c_\alpha^\ell - \frac{\sqrt{Nh_N^c}\widehat{\Delta}(\mathcal{C},x)}{\widehat{\sigma}_\ell(\mathcal{C},x)} - \sqrt{1-\widehat{\rho}^2(\mathcal{C},x)}\cdot Z_2 \le \widehat{\rho}(\mathcal{C},x)\cdot Z_1, Z_1 \le c_\alpha^u\right) \ge 1-\alpha,$$

where Z_1 and Z_2 are independent $\mathcal{N}(0,1)$.

[12] According to NASDAQ, by the end of 2010 the U.S. market share of both firms in plumbing, electrical, and kitchen products (part of the broader home improvement products category) was approximately 40%, with 23% for Home Depot and 17% for Lowe's.

pure-strategy NE behavior (Assumption 3′). We also maintain that Y_1 and Y_2 are mutual strategic substitutes w.p.1, and, therefore, Assumption 4 is satisfied with $f_p(y_{-p}) = y_{-p}$. From here, construction of the indicator functions $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y})$ and $\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y}, \mathbf{y})$ follows directly from the examples in Sections 4.3 and 6. However, for clarity, we show in Appendix C how to compute them for the specific case of this section ($P = 2$ and mutual strategic substitutes).

8.1 *Data overview*

We define a market as a *core based statistical area* (CBSA) in the contiguous United States (the 48 states that do not include Alaska or Hawaii). Each observation in our sample corresponds to a CBSA[13] and our sample consists of $N = 951$ such observations.[14] The total number of stores of firm p that existed in market i in the year 2008 is denoted by $Y_{p,i}$. Table 1 summarizes the outcomes observed in the data.

The sample correlation between Y_1 and Y_2 was 0.8475. The collection of *different* outcomes observed in the sample was 105, a number that is significantly smaller than the cardinality of the action space implied by Table 2 (the maximum number of stores by either player in a market was 61). We also included the following covariates in our analysis:[15]

$$\mathrm{INC}_i \equiv \text{income per household in } i\text{th market (dollars)},$$

$$\mathrm{POP}_i \equiv \text{Population in } i\text{th market},$$

$$D_{p,i} \equiv \text{Distance between } i\text{th market and corporate headquarters of } p.$$

The source for INC is the 2000 census; POP is the estimated CBSA population[16] in 2003. Our sample covers approximately 93% of the estimated population in the United States

TABLE 1. Overview of outcomes observed.

	No Firm Entered	Home Depot Entered	Only Home Depot Entered	Lowe's Entered	Only Lowe's Entered	Both Firms Entered
Number of markets	263	487	124	564	201	363

[13] The Office of Budget and Management defines a CBSA as an area that consists of one or more counties, and includes the counties containing the core urban area as well as any adjacent counties that have a high degree of social and economic integration (as measured by commuting to work) with the urban core.

[14] Jia (2008) defined a market as a county and allows for intrafirm complementarities across markets. Our modeling approach allows for such complementarities across counties within a single CBSA *as long the resulting decision problem can be accurately described by the game studied here.*

[15] The headquarters of Home Depot ($p = 1$) and Lowe's ($p = 2$) are located in Atlanta, GA (30339) and Mooresville, NC (28115), respectively. We computed $D_{p,i}$ as the minimum distance between the headquarters zip code of firm p, and the zip codes that conform to the ith market. The distance between both headquarters is 242 miles.

[16] The source is Table 1 of the *Annual Estimates of the Population of Metropolitan and Micropolitan Statistical Areas: April 1, 2000 to July 1, 2003* (CBSA-EST2003-01) published by the Population Division, U.S. Census Bureau.

Table 2. Summary statistics of outcomes observed[a].

	Home Depot Stores (Y_1)	Lowe's Stores (Y_2)	Total Stores ($Y_1 + Y_2$)
25th percentile	0	0	0
50th percentile	1	1	1
75th percentile	1	1	3
95th percentile	11	7	17
98th percentile	22	13	31
Maximum	61	27	88
Average	1.99	1.56	3.55
Standard deviation	5.39	2.96	8.06
Total	1899	1485	3384

[a]Existing stores in 2008.

in 2003. We maintain that $(Y_{1,i}, Y_{2,i}, \mathrm{INC}_i, \mathrm{POP}_i, D_{1,i}, D_{2,i})_{i=1}^N$ is an iid sample (to allow spatial dependence, see footnote 8). Due to the relative proximity of both headquarters (240 miles), we focus primarily on

$$D_i \equiv \min\{D_{1,i}, D_{2,i}\}.$$

An overview of the data is presented in Table 3.

Distance from headquarters as a determinant of entry was studied, for example, by Manuszak and Moul (2008) for office supply stores and by Jia (2008) and Holmes (2011) in the case of big box retailers. One argument is that proximity to markets that are otherwise unattractive may induce entry. In our case, Table 3 seems to indicate that markets without entry tend to be systematically farther away from both firms' headquarters relative to markets where at least one firm is present. The importance of population and income per household as determinants of entry also seems evident in the table.

There are many potentially interesting questions about the data that could be addressed by applying our methodology. We focus on the following issues here. First, we study some properties of equilibria where no firm enters a market as well as equilibria where both firms enter. Then we move on to different types of action profiles that were rarely observed in the data, and we ask whether this is because these are seldom NE ac-

Table 3. Summary statistics of additional covariates.

	Markets Where No Firm Entered			Markets Where at Least One Firm Entered		
	INC	POP	D	INC	POP	D
25th percentile	37,846	25,118	403	43,725	59,584	251
50th percentile	42,500	33,936	696	53,095	112,947	473
75th percentile	50,269	43,181	974	69,375	279,966	741
Average	44,651	37,590	806	62,773	382,667	631
Min	21,266	12,238	41	25,625	20,928	0
Max	100,130	150,959	2212	200,001	11,421,928	2180

tions or whether the selection mechanism avoids choosing such NE. Finally, we study whether there is evidence that the selection mechanism favors entry by either firm in a discernible way. Our analysis reveals some interesting features of the underlying structural model.

8.1.1 *Kernels and bandwidths employed* Our confidence intervals are constructed as we described in Section 7, specifically, in equations (22) and (23). For simplicity, in Section 7, we assumed the use of the same bandwidth for every continuous covariate in X, but this can be easily relaxed and all the asymptotic results shown there hold exactly as described after replacing h_N^c with $\prod_{\ell=1}^{c} h_{\ell,N}$, where $h_{\ell,N}$ is the bandwidth used for X^ℓ, the ℓth continuous covariate in X (c denotes the number of continuous covariates in X). Our bandwidths are of the form $h_{\ell,N} = 2\cdot(X^\ell_{(0.90)} - X^\ell_{(0.10)})\cdot N^{-\alpha}$, where $\alpha = 1/(c+4)+10^{-6}$ and $X^\ell_{(\tau)}$ denotes the τth quantile of X^ℓ. We use a multiplicative kernel of the form $K(\psi^1,\dots,\psi^c) = \prod_{\ell=1}^{c}\mathcal{K}(\psi^\ell)$, with $\mathcal{K}(z) = \frac{15}{16}(1-z^2)^2\mathbb{1}\{|z|\le 1\}$ (biweight kernel). Our kernel and bandwidths satisfy Assumption 5.

8.2 *On the equilibrium where no firm enters a market*

There were 263 markets (28% of our observations) where no firm entered. In addition to POP and INC, distance D seems to be a determinant of entry (see Table 3), as markets where there was no entry tend to be systematically farther away. In Table 4 we estimate 95% confidence intervals for $P_{\mathcal{E}}((0,0),X)$ (the probability that $(0,0)$ is a NE profile), and for $Q_{\mathcal{E}}((0,0),X)$ (the probability that the selection mechanism chooses a NE that includes $(0,0)$ given that such a NE exists). Unconditionally, the confidence interval for $P_{\mathcal{E}}((0,0))$ is [0.2563, 0.7652] and that for $Q_{\mathcal{E}}((0,0))$ is [0.3457, 1]. Thus, while the (unconditional) probability that $(0,0)$ is a NE action can be as high as $\approx 76\%$, the likelihood of

TABLE 4. 95% confidence intervals for $P_{\mathcal{E}}((0,0),X)$ and $Q_{\mathcal{E}}((0,0),X)$.

	$X=\mathbb{1}\{d_\ell\le D\le d_u\}$ No Additional Covariates		$X=(\mathbb{1}\{d_\ell\le D\le d_u\}$ INC = 40,000, POP = 20,000)	
	$P_{\mathcal{E}}((0,0),X)$	$Q_{\mathcal{E}}((0,0),X)$	$P_{\mathcal{E}}((0,0),X)$	$Q_{\mathcal{E}}((0,0),X)$
$d_\ell=0, d_u=300$	[0.1171, 0.7424]	[0.2024, 1]	[0.1621, 0.9437]	[0.2332, 1]
$d_\ell=300, d_u=800$	[0.2344, 0.7719]	[0.3369, 1]	[0.3609, 0.9809]	[0.4229, 1]
$d_\ell=800, d_u=1300$	[0.4544, 0.9103]	[0.5841, 1]	[0.5994, 0.9997]	[0.6859, 1]
$d_\ell=1300, d_u=2300$	[0.2557, 0.7653]	[0.4275, 1]	[0.3713, 0.9741]	[0.5027, 1]
	$X=(\mathbb{1}\{d_\ell\le D\le d_u\}$ INC = 45,000, POP = 60,000)		$X=(\mathbb{1}\{d_\ell\le D\le d_u\}$ INC = 60,000, POP = 200,000)	
	$P_{\mathcal{E}}((0,0),X)$	$Q_{\mathcal{E}}((0,0),X)$	$P_{\mathcal{E}}((0,0),X)$	$Q_{\mathcal{E}}((0,0),X)$
$d_\ell=0, d_u=300$	[0.0610, 0.8591]	[0.1387, 1]	[0.0321, 0.7265]	[0.1143, 1]
$d_\ell=300, d_u=800$	[0.2617, 0.9540]	[0.3337, 1]	[0.1830, 0.8816]	[0.2723, 1]
$d_\ell=800, d_u=1300$	[0.4365, 0.9878]	[0.5552, 1]	[0.3154, 0.9451]	[0.4747, 1]
$d_\ell=1300, d_u=2300$	[0.3281, 0.9706]	[0.4573, 1]	[0.2270, 0.9116]	[0.3819, 1]

selecting such a NE given that it exists can be as low as $\approx 35\%$. We want to investigate if and how these results change with D, with and without controlling for POP and INC.

Compared with markets that are farther away, nearby markets are consistent with a much smaller propensity to select a NE that includes $(0, 0)$ given that such a NE exists. For instance, while this propensity can be as low as $\approx 21\%$ in markets closer than 300 miles, it cannot be smaller than 58% in markets that are between 800 and 1300 miles away. For relatively unattractive markets (INC = 40,000, POP = 20,000), these figures change to 23% and 68% respectively. The lower confidence bound for $Q_{\mathscr{E}}((0, 0), X)$ increased steadily with distance up to 1300 miles, declining a little bit afterward. This cannot be explained solely by the presence of major metropolitan areas in the West Coast since it was also observed for smaller markets (INC = 40,000, POP = 20,000). Our confidence intervals for $P_{\mathscr{E}}((0, 0), X)$ suggest that $(0, 0)$ is a NE profile with high probability. For example, we cannot refute that $(0, 0)$ is a NE action with $\approx 73\%$ probability at *every* distance range studied. The fact that the proportion of markets where nobody entered was well below 73% suggests the presence of a selection mechanism that favors entry.

8.3 *Equilibria where both firms enter a market*

Both firms entered into 363 markets (38% of the observations) and each one entered with two or more stores in 163 markets (17%). Let

$$\mathcal{C}^a = \{(y_1, y_2) : 1 \leq y_p \leq 60 \text{ for } p = 1, 2\},$$
$$\mathcal{C}^b = \{(y_1, y_2) : 2 \leq y_p \leq 60 \text{ for } p = 1, 2\}.$$

We construct 95% confidence intervals for $P_{\mathscr{E}}(\mathcal{C}^a, X)$, $Q_{\mathscr{E}}(\mathcal{C}^a, X)$, $P_{\mathscr{E}}(\mathcal{C}^b, X)$, and $Q_{\mathscr{E}}(\mathcal{C}^b, X)$ next. We present results both for general NE behavior and for pure-strategy-only NE in Table 5.

Nash equilibrium in $\mathcal{C}^a$ seems to exist with very high probability in large, wealthy markets (POP $\geq$ 150,000, INC = \$80,000). The propensity to select such a NE when it exists is also remarkably high (at least $\approx 93\%$ with mixed strategies and $\approx 96\%$ if we maintain pure-strategy NE). These features are similar, but a bit weaker, for NE in $\mathcal{C}^b$. Maintaining pure-strategy NE behavior, we see that while the unconditional probability

Table 5. 95% confidence intervals.

	Mixed and Pure Strategy NE Behavior		Pure Strategy NE Behavior Only	
	$P_{\mathscr{E}}(\mathcal{C}^a, X)$	$Q_{\mathscr{E}}(\mathcal{C}^a, X)$	$P_{\mathscr{E}^*}(\mathcal{C}^a, X)$	$Q_{\mathscr{E}^*}(\mathcal{C}^a, X)$
$X = \emptyset$ (unconditional)	[0.3598, 1]	[0.3598, 1]	[0.3598, 0.4396]	[0.8948, 1]
$X = (\mathbb{1}\{\text{POP} \geq 150{,}000\}, \text{INC} = 80{,}000)$	[0.8979, 1]	[0.9359, 1]	[0.8977, 0.9975]	0.9635, 1]
	$P_{\mathscr{E}}(\mathcal{C}^b, X)$	$Q_{\mathscr{E}}(\mathcal{C}^b, X)$	$P_{\mathscr{E}^*}(\mathcal{C}^b, X)$	$Q_{\mathscr{E}^*}(\mathcal{C}^b, X)$
$X = \emptyset$ (unconditional)	[0.1544, 0.4396]	[0.3762, 1]	[0.1544, 0.2221]	[0.8036, 1]
$X = (\mathbb{1}\{\text{POP} \geq 150{,}000\}, \text{INC} = 80{,}000)$	[0.5526, 0.9974]	[0.6462, 1]	[0.5524, 0.7779]	[0.8881, 1]

TABLE 6. 95% confidence intervals. Pure-strategy NE behavior is the maintained assumption.

	$P_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{HD}}, X)$	$Q_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{HD}}, X)$	$P_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{LO}}, X)$	$Q_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{LO}}, X)$
$X = \emptyset$ (unconditional)	[0.2409, 0.7436]	[0.3350, 1]	[0.2841, 0.7332]	[0.4013, 1]
$X = \mathbb{1}\{\mathrm{POP} \le 250{,}000\}$	[0.1573, 0.6801]	[0.2467, 1]	[0.2878, 0.6801]	[0.4491, 1]
$X = \mathbb{1}\{\mathrm{POP} > 250{,}000\}$	[0.5577, 1]	[0.5977, 1]	[0.2338, 0.9692]	[0.2861, 1]
$X = \mathbb{1}\{\mathrm{INC} \le 45{,}000\}$	[0.1099, 0.5904]	[0.2290, 1]	[0.3131, 0.5904]	[0.6125, 1]
$X = \mathbb{1}\{\mathrm{INC} > 45{,}000\}$	[0.3061, 0.8428]	[0.3857, 1]	[0.2544, 0.8270]	[0.3286, 1]

that a NE in $\mathcal{C}^a$ exists can be as low as $\approx 35\%$, the likelihood of selecting such a NE when it exists is at least $\approx 90\%$. In the case of profiles in $\mathcal{C}^b$, these figures change to $\approx 15\%$ and 80%, respectively. Our results reveal a selection mechanism with a high propensity to select NE where both firms enter whenever such a NE exists.

8.4 *Does the selection mechanism favor either firm in a systematic way?*

Home Depot entered with more stores than Lowe's in 248 markets and the opposite was true in 290 markets. The median values of INC and POP in markets where Home Depot entered with more stores than Lowe's were approximately \$63,000 and 170,000, respectively. These figures dropped to \$47,000 and 83,000 for markets where Lowe's entered with more stores than Home Depot. Is the selection mechanism biased in favor of either firm depending on INC and POP? Let

$$\mathcal{C}^{\mathrm{HD}} = \{(y_1, y_2) : y_2 + 1 \le y_1 \le 60\}, \qquad \mathcal{C}^{\mathrm{LO}} = \{(y_1, y_2) : y_1 + 1 \le y_2 \le 60\}.$$

Home Depot enters with more stores than Lowe's in every profile in $\mathcal{C}^{\mathrm{HD}}$ and the opposite is true for every profile in $\mathcal{C}^{\mathrm{LO}}$. For brevity, we maintain pure-strategy NE behavior only and construct 95% confidence intervals for $P_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{HD}}, X)$, $Q_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{HD}}, X)$, $P_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{LO}}, X)$, and $Q_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{LO}}, X)$ in Table 6.

In all cases, the upper bounds for $P_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{HD}}, X)$ and $P_{\mathscr{E}^*}(\mathcal{C}^{\mathrm{LO}}, X)$ either coincide exactly or are very close to each other. This reveals a striking feature of the data. *Every* market that was compatible with the existence of a NE in $\mathcal{C}^{\mathrm{LO}}$ was *also* compatible[17] with the existence of a NE in $\mathcal{C}^{\mathrm{HD}}$. Furthermore, only 10 markets (out of 951) were compatible with equilibria in $\mathcal{C}^{\mathrm{HD}}$ but ruled out equilibria[18] in $\mathcal{C}^{\mathrm{LO}}$. These markets were among the 20 largest and wealthiest in the sample. Thus, except perhaps for these types of markets, our results support the assertion that equilibria in $\mathcal{C}^{\mathrm{HD}}$ and $\mathcal{C}^{\mathrm{LO}}$ tend to *coexist* with high probability. However, the probability of selecting NE in $\mathcal{C}^{\mathrm{HD}}$ ($\mathcal{C}^{\mathrm{LO}}$) when they exist appears to be systematically higher for larger, wealthier (smaller, less wealthy) markets. If $\mathrm{POP} \ge 250{,}000$, equilibria favoring Home Depot are selected with at least 60% probability whenever they exist. Coincidentally, this propensity is also at least $\approx 60\%$ for equilibria favoring Lowe's in markets where $\mathrm{INC} \le \$45{,}000$.

[17]That is, $\max_{\mathbf{y}\in\mathcal{C}^{\mathrm{LO}}}\{\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y}_i, \mathbf{y})\} = 1$ implied $\max_{\mathbf{y}\in\mathcal{C}^{\mathrm{HD}}}\{\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y}_i, \mathbf{y})\} = 1$ for *every market* $i = 1, \dots, 951$ *in our sample.*

[18]That is, we had $\max_{\mathbf{y}\in\mathcal{C}^{\mathrm{LO}}}\{\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y}_i, \mathbf{y})\} = 0$ and $\max_{\mathbf{y}\in\mathcal{C}^{\mathrm{HD}}}\{\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y}_i, \mathbf{y})\} = 1$ for only 10 markets i.

8.5 *Some action profiles seldom observed in the data*

Many profiles in $\mathcal{A}$ were very rarely observed as outcomes in the data. For example, 48% of all action profiles (y_1, y_2) where $\max\{y_1, y_2\} \leq 10$ (clearly a subset of $\mathcal{A}$) were *not observed as the outcome in any market in our sample*. Two instances were[19] $(5, 6)$ (Home Depot enters with five stores and Lowe's with six) and $(4, 0)$ (only Home Depot enters with four stores). Unconditionally, a 95% confidence interval for $P_{\mathscr{E}}((5, 6))$ is $[0, 0.0683]$ and for $P_{\mathscr{E}}((4, 0))$ is $[0, 0.9195]$. If we presuppose pure-strategy NE behavior, these intervals shrink to $[0, 0.0371]$ and $[0, 0.4956]$, respectively. Remarkably, while the data appear consistent with widespread existence of a NE that includes $(4, 0)$, the probability that $(5, 6)$ is a NE profile is no more than $\approx 6\%$. A question of interest is whether this indicates a more general pattern, namely, that rarely observed outcomes where both firms enter with multiple stores are absent because they are rarely NE profiles, while outcomes where only one firm enters with multiple stores are absent because the selection rule avoids such NE. Let

$$\mathcal{C}^d = \{(y_1, y_2) : 2 \leq y_p \leq 4 \text{ for } p = 1, 2\},$$

$$\mathcal{C}^e = \{(y_1, y_2) : 2 \leq y_p \leq 4 \text{ and } y_{-p} = 0 \text{ for } p = 1, 2\}.$$

Both firms enter with between two and four stores in each profile in $\mathcal{C}^d$, and only one firm enters with between two and four stores in $\mathcal{C}^e$. We observed outcomes in $\mathcal{C}^d$ in 63 markets (6.6% of our sample), and this number was 34 (3.5% of our sample) for $\mathcal{C}^e$. We construct 95% confidence intervals next; for brevity we focus on pure-strategy Nash equilibrium behavior.

Our results in Table 7 indicate that equilibria in $\mathcal{C}^d$ are rare (they exist at most 6% of the time in small markets and at most 25% of the time in large markets), and the propensity to select such equilibria when they exist is high (at least 45% in small markets and at least 58% in large, wealthy markets). Conversely, we cannot rule out widespread existence of equilibria in $\mathcal{C}^e$ (this probability can be as high as 71% in small markets and 88% in large markets), coupled with a very low propensity to select such equilibria when they exist (as low as $\approx 6\%$ in all cases studied). While the scarcity of outcomes in $\mathcal{C}^d$ appears to be due to the rarity of NE where such profiles are played, the absence of outcomes in $\mathcal{C}^e$ seems to follow from a low propensity to select NE where those profiles are played.

TABLE 7. 95% confidence intervals. Pure-strategy NE behavior is the maintained assumption.

	$P_{\mathscr{E}^*}(\mathcal{C}^d, X)$	$Q_{\mathscr{E}^*}(\mathcal{C}^d, X)$	$P_{\mathscr{E}^*}(\mathcal{C}^e, X)$	$Q_{\mathscr{E}^*}(\mathcal{C}^e, X)$
$X = \emptyset$ (unconditional)	[0.0550, 0.1345]	[0.4878, 1]	[0.0273, 0.6568]	[0.0432, 1]
$X = (\text{POP} = 50{,}000, \text{INC} = 60{,}000)$	[0.0115, 0.0640]	[0.4527, 1]	[0.0277, 0.7166]	[0.0616, 1]
$X = (\text{POP} = 150{,}000, \text{INC} = 60{,}000)$	[0.0281, 0.1032]	[0.5140, 1]	[0.0296, 0.7667]	[0.0605, 1]
$X = (\text{POP} = 300{,}000, \text{INC} = 60{,}000)$	[0.0926, 0.2552]	[0.5824, 1]	[0.0276, 0.8883]	[0.0571, 1]

[19] "Nearby" profiles $(6, 5)$ and $(4, 1)$ (and others) *were* observed as outcomes in our sample.

9. Concluding remarks

Real world data in simultaneous discrete choice games with large action spaces $\mathcal{A}$ tend to have a common feature, namely the collection of outcomes observed is usually concentrated in a relatively small subset of $\mathcal{A}$. Understanding the reason behind this phenomenon is important. Maintaining the assumption of equilibrium behavior, we addressed this question in a nonparametric setting which allows, for instance, cases where some action profiles in $\mathcal{A}$ are never played in equilibria. Aside from general restrictions about payoff functions, we presupposed that the researcher has ex ante knowledge about the direction of strategic interaction in the game; these predictions usually come from economic theory. This type of assumption was imposed previously in parametric settings, but its applicability depends on the empirical problem at hand. Our assumptions enabled us to characterize bounds for the probability that a prespecified action profile $\mathbf{y}$ is a Nash equilibrium profile. We also obtained a lower bound for the probability that the underlying equilibrium selection mechanism chooses equilibria where $\mathbf{y}$ is played in the event that such equilibria exist. Potentially, our results can enable the researcher to determine whether the absence of certain outcomes in the data is due to the fact that equilibria where they are played rarely exist or whether such equilibria are avoided by the selection mechanism. We applied our results to a model of multiple entry by the two dominant firms in the home improvement industry (Home Depot and Lowe's). Our results uncovered interesting features of the underlying structural model. Aside from our main goals, our methodology and results can also be used to test specific versions of the model with stronger (e.g., parametric) assumptions.

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