SMOOTH ERGODIC THEORY OF $\mathbb{Z}^d$-ACTIONS PART 3: PRODUCT STRUCTURE OF ENTROPY

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ABSTRACT. For a smooth action of $\mathbb{Z}^d$ preserving a Borel probability measure, we show that entropy satisfies a certain "product structure" along coarse unstable manifolds. Moreover, given two smooth $\mathbb{Z}^d$-actions—one of which is a measurable factor of the other—we show that all coarse Lyapunov exponents contributing to the entropy of the factor system are coarse Lyapunov exponents of the total system and derive an Abramov–Rohlin formula for entropy subordinated to coarse unstable manifolds.

13. STATEMENT OF RESULTS

As in Part 1, take M to be a C^∞ manifold equipped with a Borel probability measure μ. Let $\alpha\colon \mathbb{Z}^d \times M \to M$ be an action by measure-preserving, measurable transformations. We moreover assume (M,μ) and α satisfy the standing hypotheses of Section 3.1. We further assume for simplicity that μ is ergodic.

13.1. **Product structure and subadditivity of entropy.** Our first main result of is the following "product structure of entropy" formula. Recall $\hat{\mathcal{L}}$ denotes the coarse Lyapunov exponents of α with respect to μ and for $\chi \in \hat{\mathcal{L}}$, $\mathscr{W}^\chi$ is the corresponding foliation by coarse Lyapunov manifolds.

Theorem 13.1. *Let $\mathcal{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation and let η be an α-invariant measurable partition. Then for $n \in \mathbb{Z}^d$,*

$$h_\mu(\alpha(n) \mid \mathcal{F} \vee \eta) = \sum_{\{\chi\in\hat{\mathcal{L}}:\chi(n)>0\}} h_\mu(\alpha(n) \mid \mathcal{F} \vee \mathscr{W}^\chi \vee \eta).$$

In particular, we have

Corollary 13.2 (Product structure of entropy)**.**

$$h_\mu(\alpha(n)) = \sum_{\{\chi\in\hat{\mathcal{L}}:\chi(n)>0\}} h_\mu(\alpha(n) \mid \mathscr{W}^\chi). \tag{29}$$

Note that if $f\colon M \to M$ and $g\colon N \to N$ are diffeomorphisms preserving μ_1 and μ_2, respectively, then there is a natural $\mathbb{Z}^2$-action on $M \times N$ preserving $\mu_1 \times \mu_2$. In this case, (29) follows immediately from the classical Ledrappier–Young entropy formula. Our result (29) suggest that, at least at the level of entropy, an ergodic α-invariant measure behaves like a product measure along coarse Lyapunov manifolds. It would be of interest to know if the unstable conditional measures are necessarily products of conditional measures along coarse Lyapunov manifolds. In homogeneous settings considered in [EL] and [EK1, EK2], similar product structures of entropy are established by first establishing a product structure of the measure along coarse manifolds.

As a direct corollary of Theorems 13.1 and 7.7, we recover the subadditivity of entropy of $\mathbb{Z}^d$-actions first obtained in [Hu] for commuting C^2 diffeomorphisms of compact manifolds.

Theorem 13.3 (c.f. [Hu, Theorem B]). *Let $\mathcal{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation and let η be an α-invariant measurable partition. Then for all $n, m \in \mathbb{Z}^d$*

(1) $h_\mu(\alpha(n+m) \mid \mathcal{F} \vee \eta) \leq h_\mu(\alpha(n) \mid \mathcal{F} \vee \eta) + h_\mu(\alpha(m) \mid \mathcal{F} \vee \eta)$;
(2) moreover, if $\mathcal{F} \vee \mathscr{W}^u_n = \mathcal{F} \vee \mathscr{W}^u_m$ then equality holds.

We also obtain the following exact dimensionality formula for measures invariant under $\mathbb{Z}^k$-actions. Let $\mathcal{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation and let η be an α-invariant measurable partition. For $\chi \in \mathcal{L}$ let $d^{\mathcal{F},\chi,\eta}(\mu)$ be the almost surely constant value of the pointwise dimension of μ along $\mathcal{F} \vee \mathscr{W}^\chi \vee \eta$ and for $n \in \mathbb{Z}^d$ let $d^{u,\mathcal{F},\eta}_n(\mu)$ be the almost-surely constant value of the pointwise dimension of μ along $\mathcal{F} \vee \mathscr{W}^u_n \vee \eta$. From Theorems 13.1 and 7.7 we obtain

Corollary 13.4 (Product structure of unstable dimension). *For any $n \in \mathbb{Z}^d$,*

$$d^{u,\mathcal{F},\eta}_n(\mu) = \sum_{\chi(n)>0} d^{\mathcal{F},\chi,\eta}(\mu).$$

13.2. **Measurable factors and coarse Abramov–Rohlin formula.** Consider a second action $\hat{\alpha}$ of $\mathbb{Z}^d$ on (N, ν) satisfying the standing hypotheses of Section 3.1. We say that $\hat{\alpha}$ is a *measurable factor of* α if there is a measurable map $\psi \colon M \to N$ with $\psi_*\mu = \nu$ and $\psi \circ \alpha(n) = \hat{\alpha}(n) \circ \psi$ for all $n \in \mathbb{Z}^d$. Let $\mathcal{A}^\psi$ denote the α-invariant partition of (M, μ) into level sets of ψ.

We assume μ and thus ν are ergodic. To distinguish data associated to each action, let $\hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$ and $\hat{\mathcal{L}}^{\alpha}(\mu)$ denote, respectively, the coarse Lyapunov exponents for the actions $\hat{\alpha}$ and α on (N, ν) and (M, μ).

Consider a coarse Lyapunov exponent $\hat{\chi} \in \hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$ of $\hat{\alpha}$ and suppose that

$$h_\nu(\hat{\alpha}(n) \mid \mathscr{W}^{\hat{\chi}}) > 0 \tag{30}$$

for some $n \in \mathbb{Z}^d$ with $\hat{\chi}(n) > 0$. Let $E = \ker(\hat{\chi}) \subset \mathbb{R}^d$ be the *Lyapunov hyperplane* determined by $\hat{\chi}$. It follows from Corollary 13.2 and (30) that for any open cone $C \subset \mathbb{R}^d$ containing E, the function

$$n \mapsto h_\nu(\hat{\alpha}(n))$$

is not a linear function on $C \cap \mathbb{Z}^d$. By the classical Abramov-Rohlin formula (19), for every $n \in \mathbb{Z}^d$ we have that

$$h_\nu(\hat{\alpha}(n)) = h_\mu(\alpha(n)) - h_\mu(\alpha(n) \mid \mathcal{A}^\psi).$$

If no Lyapunov exponent of α were proportional to $\hat{\chi}$ then, taking any open cone $C' \subset C \subset \mathbb{R}^d$ containing E and disjoint from the kernels of all non-zero Lyapunov exponents in $\mathcal{L}^\alpha(\mu)$ it follows from Theorems 13.1 and 7.7 that both $h_\mu(\alpha(n))$ and $h_\mu(\alpha(n) \mid \mathcal{A}^\psi)$ coincide with linear functions on $C' \cap \mathbb{Z}^d$ contradicting the choice of C above.

It thus follows that every coarse Lyapunov exponent $\hat{\chi} \in \hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$ that contributes entropy to $\hat{\alpha}$ is proportional to a Lyapunov exponent of α. We say $\chi \in \hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$ is *essential* if

$$h_\nu(\hat{\alpha}(n) \mid \mathscr{W}^\chi) > 0$$

for some (and hence all) $n \in \mathbb{Z}^d$ with $\chi(n) > 0$. Let $\hat{\mathcal{L}}^{\hat{\alpha}}_{\text{ess}}(\nu)$ denote the essential coarse Lyapunov exponents of the actions of $\hat{\alpha}$ on (N, ν). As remarked above, all essential exponents $\hat{\chi}$ of $\hat{\alpha}$ are proportional to Lyapunov exponents of α. We show that they are, in fact, positively proportional.

Theorem 13.5. *We have $\hat{\mathcal{L}}^{\hat\alpha}_{\mathrm{ess}}(\nu) \subset \hat{\mathcal{L}}^{\alpha}(\mu)$.*

Analogous statements to Theorem 13.5 are established (for all coarse Lyapunov exponents) in [KRH, Section 6.2] and [KK, Lemma 2.3] under the assumption that the factor map ψ is Hölder continuous using the exponential contraction along stable manifolds. Our more general statement in Theorem 13.5 follows using only entropy considerations and Theorem 7.7.

Given $\chi \in \hat{\mathcal{L}}^{\alpha}(\mu)$ let $\hat\chi \in \hat{\mathcal{L}}^{\hat\alpha}(\nu)$ denote the equivalence class of exponents positively proportional to those in χ if such a class exists; if no such class exists let $\hat\chi$ be the 0 linear functional. If $\hat\chi = 0$, let $\mathscr{W}^{\hat\chi}$ denote the point partition on (N, ν).

Recall the classical Abramov–Rohlin formula (19). We establish an analogous formula for entropy subordinate to coarse Lyapunov foliations under a measurable factor map ψ between smooth $\mathbb{Z}^d$-actions α and $\hat\alpha$.

Theorem 13.6 (Coarse Abramov–Rohlin formula)**.** *Let $\chi \in \hat{\mathcal{L}}^{\alpha}(\mu)$. Then for $n \in \mathbb{Z}^d$ with $\chi(n) > 0$ we have*

$$h_\mu(\alpha(n) \mid \mathscr{W}^{\chi}) = h_\nu(\hat\alpha(n) \mid \mathscr{W}^{\hat\chi}) + h_\mu(\alpha(n) \mid \mathscr{W}^{\chi} \vee \mathcal{A}^{\psi}).$$

We note that in Proposition 3.1 and Corollary 3.4 of [EL], an analogous result is derived in the context of joinings of homogeneous actions in which case the factor maps are smooth.

14. Preliminaries

Recall we take M to be a C^∞ manifold equipped with a Borel probability μ and $\alpha\colon \mathbb{Z}^d \times M \to M$ an action satisfying the hypotheses of Section 3.1 with μ ergodic. $\mathcal{L}$ denotes the Lyapunov exponents of α with respect to μ.

14.1. Lyapunov hyperplanes, Weyl chambers, and complete classes of exponents.

By extending each λ_i to a linear function $\lambda_i\colon \mathbb{R}^d \to \mathbb{R}$, the *Lyapunov hyperplane* associated to a non-zero $\lambda_i \in \mathcal{L}$ is the kernel in $\mathbb{R}^d$ of λ_i. Note that if λ_i and λ_j are coarsely equivalent then λ_i and λ_j induce the same Lyapunov hyperplane.

Recall that relative to a fixed enumeration of $\lambda_i \in \mathcal{L}$, for each $n \in \mathbb{Z}^d$ we fix a permutation as in Section 4.2 so that $\lambda_{\sigma(n)(1)}(n) \geq \lambda_{\sigma(n)(2)}(n) \geq \cdots \geq \lambda_{\sigma(n)(\ell)}(n)$. We say $n \in \mathbb{Z}^d$ is *generic* if the above inequalities are all strict and n is not contained in a Lyapunov hyperplane. A (open) *Weyl chamber* is a connected component of the complement of all Lyapunov hyperplanes in $\mathbb{R}^d$. We say a non-zero $\chi \in \hat{\mathcal{L}}$ is in the *wall* of a Weyl chamber W if the boundary of W contains an open subset of the kernel of χ. A (open) *subchamber* of a Weyl chamber W is a maximal collection of generic $n \in W$ on which the permutation $n \mapsto \sigma(n)$ is constant. As Weyl chambers are open, every Weyl chamber contains a spanning set of generic $n \in \mathbb{Z}^d$. Similarly, subchambers of Weyl chambers contain spanning sets of $n \in \mathbb{Z}^d$.

Given a subset $\mathcal{I} \subset \mathcal{L}^{\alpha}(\mu)$, let $C(\mathcal{I})$ denote the *positive cone* of $\mathcal{I}$:

$$C(\mathcal{I}) := \{n \in \mathbb{Z}^d : \lambda(n) > 0 \text{ for all } \lambda \in \mathcal{I}\}.$$

More generally, if $\mathcal{F}$ is an α-invariant, $C^{1+\beta}$-tame, measurable foliation, the *positive cone* of $\mathcal{F}$ is

$$C(\mathcal{F}) := \{n \in \mathbb{Z}^d : \mathcal{F} \text{ is expanding for } \alpha(n)\}$$

where $\mathcal{F}$ is *expanding* for $\alpha(n)$ if $\mathcal{F} \mathrel{\mathring{=}} \mathcal{F} \vee \mathscr{W}^u_n$. Note that if $0 \neq \chi \in \hat{\mathcal{L}}$ then $C(\chi)$ is an open half-space called the *Lyapunov half-space* associated to χ.

As non-empty positive cones contain Weyl chambers, we have the following.

Claim 14.1. *For any $\mathcal{I}$ either $C(\mathcal{I})$ is empty or contains a spanning set of generic $n \in \mathbb{Z}^d$. The same is true for $C(\mathcal{F})$ for any α-invariant, $C^{1+\beta}$-tame, measurable foliation.*

Let $\mathfrak{I}$ denote the set of all subsets of $\mathcal{L}$ with the following property: $\mathcal{I} \in \mathfrak{I}$ if

$$\text{for any } \lambda \in \mathcal{L} \text{ such that } \lambda(n) > 0 \text{ for all } n \in C(\mathcal{I}) \text{ we have } \lambda \in \mathcal{I}.$$

We call such $\mathcal{I} \in \mathfrak{I}$ a *complete class of exponents*. Note that the subsets $\mathcal{I} \in \mathfrak{I}$ are saturated by coarse equivalence classes of Lyapunov exponents. From Proposition 4.5, given $\mathcal{I} \in \mathfrak{I}$ there exists a unique, α-invariant, $C^{1+\beta}$-tame, measurable foliation $\mathscr{W}^{\mathcal{I}}$ with $\mathscr{W}^{\mathcal{I}}(x)$ tangent to $\bigoplus_{\lambda\in\mathcal{I}} E_\lambda(x)$ at almost every x. Moreover, for $\mathcal{I} \in \mathfrak{I}$, we have $\mathscr{W}^{\mathcal{I}} = \bigvee \mathscr{W}^u_{n_j}$ for some finite set of $n_j \in C(\mathcal{I})$.

14.2. Adjacency and sufficient subsets of $\mathbb{Z}^d$.

Definition 14.2. We say two non-zero coarse Lyapunov exponents $\chi, \chi' \in \hat{\mathcal{L}}$ are *adjacent* if there exist generic n_1 and n_2 in $\mathbb{Z}^d$ such that

(1) $\sigma(n_1) \neq \sigma(n_2)$ and the permutations differ by pre-composition by disjoint adjacent transpositions: there are $1 \leq i_1 \leq i_1 + 2 \leq i_2 \leq \cdots \leq i_{\ell-1} \leq i_{\ell-1} + 2 \leq i_\ell \leq p - 1$ such that
$$\sigma(n_1) = \sigma(n_2) \circ \sigma_{i_1,i_1+1} \circ \sigma_{i_2,i_2+1} \circ \ldots \sigma_{i_\ell,i_\ell+1};$$
(2) $\{\lambda_{\sigma(n_1)(i_1)}, \ldots, \lambda_{\sigma(n_1)(i_\ell)}\} \subset \chi$ and $\{\lambda_{\sigma(n_1)(i_1+1)}, \ldots, \lambda_{\sigma(n_1)(i_\ell+1)}\} \subset \chi'$;
(3) $\{\lambda_{\sigma(n_2)(i_1)}, \ldots, \lambda_{\sigma(n_2)(i_\ell)}\} \subset \chi'$ and $\{\lambda_{\sigma(n_2)(i_1+1)}, \ldots, \lambda_{\sigma(n_2)(i_\ell+1)}\} \subset \chi$;
(4) $\operatorname{sgn}(\overline{\chi}(n_1)) = \operatorname{sgn}(\overline{\chi}(n_2))$ for all $\overline{\chi} \in \hat{\mathcal{L}}$.

Note that if (1)–(3) of Definition 14.2 hold then for $1 \leq k \leq \ell$ we have $\lambda_{\sigma(n_1)(i_k+1)} = \lambda_{\sigma(n_2)(i_k)}$ and $\lambda_{\sigma(n_1)(i_k)} = \lambda_{\sigma(n_2)(i_k+1)}$.

Definition 14.3. We say two generic n_1 and n_2 in $\mathbb{Z}^d$ are *adjacent* if either

(a) $\sigma(n_1) = \sigma(n_2)$ and $\operatorname{sgn}(\chi(n_1)) = \operatorname{sgn}(\chi(n_2))$ for all $\chi \in \hat{\mathcal{L}}$; or
(b) there exist non-zero, adjacent $\chi, \chi' \in \hat{\mathcal{L}}$ so that (1)–(3) of Definition 14.2 hold with n_1 and n_2;
(c) there is a non-zero χ with
   (a) either $\chi(n_1) < 0 < \chi(n_2)$ or $\chi(n_2) < 0 < \chi(n_1)$,
   (b) $\operatorname{sgn}(\chi'(n_1)) = \operatorname{sgn}(\chi'(n_2))$ for all $\chi' \neq \chi$,
   (c) and $\sigma(n_1)^{-1}(i) = \sigma(n_2)^{-1}(i)$ for every $\lambda_i \neq 0$ with $\lambda_i \notin \chi$.

Note in condition (c) of Definition 14.3 that n_1 and n_2 are necessarily in disjoint Weyl chambers W_1 and W_2 and $\operatorname{sgn}(\chi'(n_1)) = \operatorname{sgn}(\chi(n_2))$ for all $\chi' \neq \chi \in \hat{\mathcal{L}}$.

Definition 14.4. We say two Weyl chambers W_1 and W_2 are *adjacent* if there is a χ in the wall of W_1 and W_2 and n_1 and n_2 such that (c) of Definition 14.3 holds.

To emphasize the role of χ in Definition 14.4 we sometimes say W_1 and W_2 are *adjacent through* χ.

Given $n \in \mathbb{Z}^d$, we say that $\chi(n) < \chi'(n)$ if
$$\lambda(n) < \lambda'(n)$$
for all $\lambda \in \chi$ and $\lambda' \in \chi'$.

Definition 14.5. A subset $S \subset \mathbb{Z}^d$ is *sufficient* in a Weyl chamber W if

(1) every $n \in S$ is generic and S contains a spanning set of $\mathbb{R}^d$ in every subchamber of W;

(2) for every $n,m \in S\cap W$ there is a sequence $n = n_0, n_1, \ldots, n_\ell = m$ with $n_i \in S\cap W$ and such that n_{i-1} and n_i are adjacent for $1 \le i \le \ell$;
(3) for every non-zero χ in the wall of W, there is a $n \in S\cap W$ such that, for every $\chi' \in \hat{\mathcal{L}}$ with $\chi' \neq \chi$,
  - if $\chi(n) > 0$ then either $\chi'(n) \le 0$ or $\chi(n) < \chi'(n)$;
  - if $\chi(n) < 0$ then either $\chi'(n) \ge 0$ or $\chi(n) > \chi'(n)$.

Definition 14.6. A subset $S \subset \mathbb{Z}^d$ is *sufficient* in a collection $W_1, W_2, \ldots, W_\ell$ of Weyl chambers if

(1) for every $n,m \in S$ there is a sequence $n = n_0, n_1, \ldots, n_\ell = m$ with $n_i \in S$ and and such that n_{i-1} and n_i are adjacent for $1 \le i \le \ell$;
(2) S is sufficient in each W_i.

We have the following claim.

Lemma 14.7. *Let $\mathcal{I} \in \mathfrak{I}$ be a complete family of Lyapunov exponents. Then there exists a finite set S that is sufficient in $C(\mathcal{I})$.*

14.3. **Increasing partitions subordinate to expanding foliation.** Let $\mathcal{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation. Following Remark 8.3 of Section 8.2, for each $n \in C(\mathcal{F})$, there is a measurable parition $\xi^{\mathcal{F}}_n$ of (M,μ) subordinate to $\mathcal{F}$ and increasing for $\alpha(n)$. By adapting the construction in Remark 8.3, as in [Hu, Section 8] we obtain the following.

Proposition 14.8. *Let $n_1, \ldots, n_\ell \in C(\mathcal{F})$. Then there exists a measurable partition $\xi^{\mathcal{F}}$ of (M,μ) with*

(1) $\xi^{\mathcal{F}}$ subordinate to $\mathcal{F}$;
(2) $\alpha(n_i)\xi^{\mathcal{F}} \le \xi^{\mathcal{F}}$ for $i = 1, \ldots, \ell$;
(3) $\xi^{\mathcal{F}}$ generates for $\alpha(n_i)$; that is $\bigvee_{k=0}^{\infty} \alpha(-kn_i)\xi^{\mathcal{F}}$ is the point partition.

Note that if $\alpha(n)\xi^{\mathcal{F}} \le \xi^{\mathcal{F}}$ and $\alpha(m)\xi^{\mathcal{F}} \le \xi^{\mathcal{F}}$ then
$$\alpha(n+m)\xi^{\mathcal{F}} \le \xi^{\mathcal{F}}.$$

14.4. **Linearity of entropy on positive cones.** We have the following adaptation of [Hu, Proposition 9.1].

Lemma 14.9. *Let $\mathcal{F}$ be an α-invariant, $C^{1+\beta}$-tame, measurable foliation. Let $n_1, n_2 \in C(\mathcal{F})$ and let η be a measurable partition that is increasing for $\alpha(n_1)$ and $\alpha(n_2)$. Then*
$$h_\mu(\alpha(n_1+n_2) \mid \mathcal{F}\vee\eta) = h_\mu(\alpha(n_2) \mid \mathcal{F}\vee\eta) + h_\mu(\alpha(n_1) \mid \mathcal{F}\vee\eta).$$
In particular, if η is α-invariant then
$$n \mapsto h_\mu(\alpha(n) \mid \mathcal{F}\vee\eta)$$
coincides on $C(\mathcal{F})$ with a linear function.

Proof. Let $n_1, n_2 \in C(\mathcal{F})$. Take a measurable partition $\xi^{\mathcal{F}}$ of (M,μ) with $\xi^{\mathcal{F}}$ subordinate to $\mathcal{F}$ and increasing for $\alpha(n_1)$ and $\alpha(n_2)$ as in Proposition 14.8. It follows from Claim 12.1 that
$$h_\mu(\alpha(m) \mid \mathcal{F}\vee\eta) = H_\mu(\alpha(-m)(\xi^{\mathcal{F}}\vee\eta) \mid \xi^{\mathcal{F}}\vee\eta)$$

for $m = n_1, n_2$ and $m = n_1 + n_2$. Moreover

$$
\begin{aligned}
h_\mu(\alpha(n_1+n_2) \mid \mathcal{F}\vee\eta) &= H_\mu(\alpha(-n_1-n_2)(\xi^{\mathcal{F}}\vee\eta) \mid \xi^{\mathcal{F}}\vee\eta)\\
&= H_\mu(\alpha(-n_1)(\alpha(-n_2)(\xi^{\mathcal{F}}\vee\eta))\vee(\alpha(-n_2)(\xi^{\mathcal{F}}\vee\eta)) \mid \xi^{\mathcal{F}}\vee\eta)\\
&= H_\mu(\alpha(-n_2)(\xi^{\mathcal{F}}\vee\eta) \mid \xi^{\mathcal{F}}\vee\eta) + H_\mu(\alpha(-n_1)(\alpha(-n_2)(\xi^{\mathcal{F}}\vee\eta)) \mid \alpha(-n_2)(\xi^{\mathcal{F}}\vee\eta))\\
&= h_\mu(\alpha(n_2) \mid \mathcal{F}\vee\eta) + h_\mu(\alpha(n_1) \mid \mathcal{F}\vee\eta). \qquad \square
\end{aligned}
$$

15. Key proposition and proof of Theorem 13.1

15.1. Conditional dimensions and dependence on coarse conditional measures. Consider an α-invariant, $C^{1+\beta}$-tame, measurable foliation $\mathcal{F}$ and a complete set $\mathcal{I} \in \mathfrak{I}$ of Lyapunov exponents. According to Proposition 4.5, for $n \in C(\mathcal{I})$ we obtain a filtration by α-invariant, $C^{1+\beta}$-tame, measurable foliations

$$\{x\} \subset \mathcal{F}_n^{1,\mathcal{I}}(x) \subset \mathcal{F}_n^{2,\mathcal{I}}(x) \subset \cdots \subset \mathcal{F}_n^{u(n),\mathcal{I}}(x) := \mathcal{F}^{\mathcal{I}}(x) \tag{31}$$

where $\mathcal{F}^{\mathcal{I}} = \mathcal{F}\vee\mathscr{W}^{\mathcal{I}}$ and $\mathcal{F}_n^{j,\mathcal{I}} := \mathscr{W}_n^{u,j}\vee\mathcal{F}^{\mathcal{I}}$.

For our main proposition, fix $\mathcal{I}\in\mathfrak{I}$, let S be a sufficient set in $C(\mathcal{I})$, and let η be a measurable partition such that $\alpha(n)\eta \le \eta$ for all $n \in S$. Consider $n \in S\cap C(\mathcal{I})$. As n is generic, we have $\lambda_i(n) \neq \lambda_j(n)$ for all $\lambda_i \neq \lambda_j \in \mathcal{L}$. For each $1 \le j \le \ell$ let $\xi_n^{j,\mathcal{I},\mathcal{F}}$ be a measurable partition of (M,μ) subordinate to $\mathcal{F}_n^{j,\mathcal{I}}$ as in Remark 8.3. We write

$$\dim_n^j(\mu \mid \mathcal{F}\vee\eta \mid \mathcal{I}) := \lim \frac{\log(\mu_x^{\xi_n^{j,\mathcal{I},\mathcal{F}}\vee\eta}(B(x,r)))}{\log(r)}.$$

The limit exists by Proposition 7.4 as $\xi_n^{j,\mathcal{I},\mathcal{F}}\vee\eta$ is increasing for $\alpha(n)$. With $\dim_n^0(\mu \mid \mathcal{F}\vee\eta \mid \mathcal{I}) := 0$, for $n\in S\cap C(\mathcal{I})$ and λ_i such that $\lambda_i(n) > 0$ we write

$$\gamma_n(\lambda_i \mid \mathcal{F}\vee\eta\mid\mathcal{I}) := \dim_n^{\sigma(n)^{-1}(i)}(\mu\mid\mathcal{F}\vee\eta\mid\mathcal{I}) - \dim_n^{\sigma(n)^{-1}(i)-1}(\mu\mid\mathcal{F}\vee\eta\mid\mathcal{I}).$$

From Proposition 7.6 we have the following observation.

Claim 15.1. *If $\mathcal{I}' \subset \mathfrak{I}$ satisfies $\mathcal{I}'\subset\mathcal{I}$ and if $\hat\eta$ is a measurable partition with $\eta\le\hat\eta$ and $\alpha(n)\hat\eta\le\hat\eta$ for all $n\in S$ then for $n\in S$ and all λ_i with $\lambda_i(n)>0$*

$$\gamma_n(\lambda_i\mid\mathcal{F}\vee\hat\eta\mid\mathcal{I}') \le \gamma_n(\lambda_i\mid\mathcal{F}\vee\eta\mid\mathcal{I}).$$

Proof. Take $\xi_n^{\mathcal{I},\mathcal{F}}$ and $\xi_n^{\mathcal{I}',\mathcal{F}}$ with $\xi_n^{\mathcal{I},\mathcal{F}} \le \xi_n^{\mathcal{I}',\mathcal{F}}$ to be measurable partitions of (M,μ) subordinated to $\mathscr{W}^{\mathcal{I}\vee\mathcal{F}}$ and $\mathscr{W}^{\mathcal{I}'\vee\mathcal{F}}$, respectively, with $\alpha(n)\xi_n^{\mathcal{I},\mathcal{F}}\le\xi_n^{\mathcal{I},\mathcal{F}}$ and $\alpha(n)\xi^{\mathcal{I}',\mathcal{F}}\le\xi^{\mathcal{I}',\mathcal{F}}$ as in Remark 8.3. Then

$$\hat\eta\vee\xi^{\mathcal{I}',\mathcal{F}} \ge \eta\vee\xi^{\mathcal{I},\mathcal{F}}$$

whence by Proposition 7.6 we have

$$\gamma_n(\lambda_i\mid\mathcal{F}\vee\hat\eta\mid\mathcal{I}') \le \gamma_n(\lambda_i\mid\mathcal{F}\vee\eta\mid\mathcal{I}). \quad \square$$

We show that often the inequality in Claim 15.1 is an equality as the element n and set $\mathcal{I}$ varies.

Proposition 15.2. *Fix $\mathcal{I}\in\mathfrak{I}$. Let S be a sufficient set in $C(\mathcal{I})$ and suppose η is measurable partition such that $\alpha(n)\eta\le\eta$ for all $n\in S$. Then*

(a) for every Weyl chamber $W\subset C(\mathcal{I})$ and $n_1, n_2 \in W\cap S$ we have

$$\gamma_{n_1}(\lambda_i\mid\mathcal{F}\vee\eta\mid\mathcal{I}) = \gamma_{n_2}(\lambda_i\mid\mathcal{F}\vee\eta\mid\mathcal{I})$$

for all λ_i with $\lambda_i(n_1)>0$;

(b) *for every pair of adjacent Weyl chambers $W_1, W_2 \subset C(\mathcal{I})$ and $n_i \in W_i \cap S$ we have*

$$\gamma_{n_1}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \gamma_{n_2}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I})$$

for all λ_i with $\lambda_i(n_1) > 0$ and $\lambda_i(n_2) > 0$;

(c) *for every Weyl chamber $W \subset C(\mathcal{I})$, χ' in the wall of W, and $n \in W \cap S$ we have*

$$\gamma_n(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \gamma_n(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I} \smallsetminus \chi')$$

for all $\lambda_i \notin \chi'$ with $\lambda_i(n) > 0$.

From Proposition 15.2, for $\lambda_i \in \mathcal{I}$ we may write

$$\gamma(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I}) := \gamma_n(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I})$$

where n is any choice of $n \in S \cap C(\mathcal{I})$.

Note that in the case that η is α-invariant we may take the set S in Proposition 15.2 to be the collection of all generic elements in which case $\gamma_n(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I})$ is defined for every generic $n \in C(\mathcal{I})$. In this case we obtain that for $\lambda_i \in \mathcal{I}$, the number $\gamma(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I})$ depends only on the conditional measures along the coarse manifolds $\mathscr{W}^\chi$ where χ is the coarse Lyapunov exponent containing λ_i.

Corollary 15.3. *Let η be α-invariant. Given $\mathcal{I} \in \mathfrak{I}$ with $C(\mathcal{I}) \neq \varnothing$, any coarse exponent $\chi \subset \mathcal{I}$, and $\lambda_i \in \chi$ we have*

$$\gamma(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \gamma(\lambda_i \mid \mathcal{F} \vee \eta \mid \chi).$$

15.2. **Proof of Theorem 13.1.** Theorem 13.1 follows immediately from the above.

Proof of Theorem 13.1. First consider a generic $n \in \mathbb{Z}^d$. Let $\mathcal{U}(n) := \{\lambda_i \in \mathcal{L} : \lambda_i(n) > 0\}$ be the collection of positive Lyapunov exponents of n. We have $\mathcal{U}(n) \in \mathfrak{I}$. From Theorem 7.7, with $f = \alpha(n)$ we have that

$$h_\mu(\alpha(n) \mid \mathcal{F} \vee \eta) = \sum_{\lambda_i(n)>0} \gamma_n(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{U}(n))\lambda_i(n).$$

On the other hand, for any coarse Lyapunov exponent $\chi \in \hat{\mathcal{L}}$ with $\chi(n) > 0$, we have (again from Theorem 7.7) that

$$h_\mu(\alpha(n) \mid \mathcal{F} \vee \mathscr{W}^\chi \vee \eta) = \sum_{\lambda_i \in \chi} \gamma_n(\lambda_i \mid \mathcal{F} \vee \eta \mid \chi)\lambda_i(n).$$

The result then follows for all generic n by Corollary 15.3.

Now, consider a non-generic $n \in \mathbb{Z}^d$. From Claim 12.1 we have

$$h_\mu(\alpha(n) \mid \mathcal{F} \vee \eta) = h_\mu(\alpha(n) \mid \mathcal{F} \vee \mathscr{W}^u_n \vee \eta).$$

Take $\hat{\mathcal{F}} = \mathcal{F} \vee \mathscr{W}^u_n$. Then $\hat{\mathcal{F}}$ is expanding for $\alpha(n)$.

Since $C(\hat{\mathcal{F}}) \neq \varnothing$, it follows that $C(\hat{\mathcal{F}})$ contains a spanning set of generic points. From Lemma 14.9, the functions

$$n \mapsto h_\mu(\alpha(n) \mid \hat{\mathcal{F}} \vee \eta)$$

and

$$n \mapsto \sum_{\{\chi \in \hat{\mathcal{L}} : \chi(n)>0\}} h_\mu(\alpha(n) \mid \hat{\mathcal{F}} \vee \mathscr{W}^\chi \vee \eta)$$

extend from $C(\hat{\mathcal{F}})$ to linear functions on $\mathbb{R}^d$ and coincide on a spanning set. It follows that they agree on $C(\hat{\mathcal{F}})$. As $\hat{\mathcal{F}} \vee \mathscr{W}^\chi = \mathcal{F} \vee \mathscr{W}^\chi$ for all $\chi \in \hat{\mathcal{L}}$ with $\chi(n) > 0$, the result follows for n. $\square$

15.3. **Proof of Proposition 15.2 and Corollary 15.3.** We establish the proposition and corollary, completing the proof of Theorem 13.1.

Proof of Proposition 15.2. Fix $\mathcal{I} \in \mathfrak{I}$. Fix a Weyl chamber $W \subset C(\mathcal{I})$ and consider $n_1, n_2 \in W \cap S$. We prove (a) of Proposition 15.2 by induction on the index of the intermediate foliations in the filtration (31) corresponding to n_1 and n_2. Note for $\lambda_i \in \mathcal{L}$, $\lambda_i(n_1) > 0$ if and only if $\lambda_i(n_2) > 0$.

It follows immediately from the geometric definition of the transverse dimensions that if $\lambda_i(n_1) > 0$ then

$$\gamma_{n_1}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \gamma_{n_2}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I})$$

whenever $\sigma(n_1) = \sigma(n_2)$. Indeed in this case the intermediate foliations in (31) for n_1 and n_2 coincide.

If $\sigma(n_1) \neq \sigma(n_2)$ then, as any $n_1, n_2 \in W \cap S$ may be joined by a string of adjacent $n_i \in W \cap S$, it is therefore enough to verify the proposition in the case that n_1 and n_2 are adjacent. As no exponent changes sign in W, we may thus assume n_1 and n_2 satisfy (b) of Definition 14.3; that is, we assume there are adjacent χ, χ' so that (1)–(3) of Definition 14.2 hold for n_1 and n_2. Let $i_1, \dots, i_\ell$ be as in (1)–(3) of Definition 14.2.

Consider any $1 \leq j$. Assume by induction that we have shown

$$\gamma_{n_1}(\lambda_{\sigma(n_1)(m)} \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \gamma_{n_2}(\lambda_{\sigma(n_2)(m)} \mid \mathcal{F} \vee \eta \mid \mathcal{I})$$

for all $m < j$. Observe

Claim 15.4. *If neither $j \neq i_k$ nor $j \neq i_{k+1}$ then*

$$\lambda_{n_1}(\lambda_{\sigma(n_1)(j)} \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \lambda_{n_2}(\lambda_{\sigma(n_2)(j)} \mid \mathcal{F} \vee \eta \mid \mathcal{I}).$$

Indeed, if neither $j \neq i_k$ nor $j \neq i_{k+1}$ then $\mathcal{F}^{j,\mathcal{I}}_{n_1} = \mathcal{F}^{j,\mathcal{I}}_{n_2}$ and $\mathcal{F}^{j-1,\mathcal{I}}_{n_1} = \mathcal{F}^{j-1,\mathcal{I}}_{n_2}$ whence equality of transverse dimensions follows from definition.

Consider now the case $j = i_k$. Fix $i \neq i'$ with $\sigma(n_1)(i_k) = i$ and $\sigma(n_2)(i_k) = i'$. Then $\sigma(n_1)(i_k + 1) = i'$ and $\sigma(n_2)(i_k + 1) = i$. We show simultaneously the equalities

$$\begin{aligned} \gamma_{n_1}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I}) &= \gamma_{n_2}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I}), \\ \gamma_{n_1}(\lambda_{i'} \mid \mathcal{F} \vee \eta \mid \mathcal{I}) &= \gamma_{n_2}(\lambda_{i'} \mid \mathcal{F} \vee \eta \mid \mathcal{I}). \end{aligned} \tag{32}$$

Part (a) of the proposition then follows from induction on j and Claim 15.4.

To establish (32), take

$$\hat{\mathcal{F}} = \mathcal{F} \vee \mathscr{W}^{u,i_{k+1}}_{n_1} = \mathcal{F} \vee \mathscr{W}^{u,i_{k+1}}_{n_2}$$

and

$$\widetilde{\mathcal{F}} = \mathcal{F} \vee \mathscr{W}^{u,i_{k-1}}_{n_1} = \mathcal{F} \vee \mathscr{W}^{u,i_{k-1}}_{n_2}.$$

We have that

$$h_\mu(\cdot \mid \hat{\mathcal{F}} \vee \eta) - h_\mu(\cdot \mid \widetilde{\mathcal{F}} \vee \eta)$$

extends from $S \cap W$ to a linear function L on $\mathbb{R}^d$. Moreover, as S contains a spanning set in each subchamber of W, on $S \cap W$ the function L coincides simultaneously with the linear functions given by the expressions

- $n \mapsto \gamma_{n_1}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I})\lambda_i(n) + \gamma_{n_1}(\lambda_{i'} \mid \mathcal{F} \vee \eta \mid \mathcal{I})\lambda_{i'}(n)$
- $n \mapsto \gamma_{n_2}(\lambda_{i'} \mid \mathcal{F} \vee \eta \mid \mathcal{I})\lambda_{i'}(n) + \gamma_{n_2}(\lambda_i \mid \mathcal{F} \vee \eta \mid \mathcal{I})\lambda_i(n)$.

As λ_i and $\lambda_{i'}$ are linearly independent and as $S \cap W$ spans $\mathbb{R}^d$, (32) and conclusion (a) follow.

For (b) consider adjacent W_1 and W_2 in $C(\mathcal{I})$. Let χ be as in Definition 14.4 and take $n_i \in W_i \cap S$ satisfying (c) of Definition 14.3 with $\chi(n_1) > 0 > \chi(n_2)$. It follows that

$\chi \notin \mathcal{I}$. Let

$\ell := \max\{j : \lambda_{\sigma(n_1)(j)} \in \mathcal{U}(n_1) \smallsetminus \chi\} = \max\{j : \lambda_{\sigma(n_2)(j)}(n_2) > 0 \text{ and } \lambda_{\sigma(n_1)(j)}(n_1) > 0\}$.

Since $\sigma(n_1)(j) = \sigma(n_2)(j)$ for all $1 \leq j \leq \ell$ it follows that the filtrations (31) corresponding to n_1 and n_2 coincide for all $1 \leq j \leq \ell$ whence the transverse dimensions coincide for all $\lambda_i \notin \chi$ with $\lambda_i(n_1) > 0$; that is, for all λ_i with $\lambda_{\sigma(n_2)(j)}(n_2) > 0$ and $\lambda_{\sigma(n_1)(j)}(n_1) > 0$.

For (c), take χ' in the wall of W and $n \in S \cap W$ with

$$0 < \lambda'(n) < \lambda(n) \text{ for all } \lambda' \in \chi' \text{ and } \lambda \notin \chi' \text{ with } \lambda(n) > 0. \tag{33}$$

Let

$$\ell = \max\{j : \lambda_{\sigma(n)(j)} \in \mathcal{U}(n) \smallsetminus \chi'\}.$$

Then $\{\lambda_{\sigma(n)(j)} : 1 \leq j \leq \ell\} \cap \mathcal{I} = \mathcal{I} \smallsetminus \chi'$. From (33) we have

$$\mathcal{F}_n^{j,\mathcal{I}}(x) = \mathcal{F}_n^{j,\mathcal{I}\smallsetminus\chi'}(x)$$

for all $1 \leq j \leq \ell$. The equality of transverse dimensions follows from definition. □

We finish with the proof of Corollary 15.3. Recall we now assume η is α-invariant whence for $\mathcal{I} \in \mathfrak{I}$ and $\lambda_i \in \mathcal{I}$ the dimension $\gamma_n(\lambda_j \mid \mathcal{F} \vee \eta \mid \mathcal{I})$ is defined for every generic $n \in C(\mathcal{I})$ and independent of $n \in C(\mathcal{I})$.

Proof of Corollary 15.3. Given $\mathcal{I} \in \mathfrak{I}$ and $\lambda_i \in \mathcal{I}$ let $\chi \in \hat{\mathcal{L}}$ be the coarse Lyapunov exponent containing λ_i. Let $\chi' \subset \mathcal{I}$ be such that $\chi' \neq \chi$ and χ' is in the wall of some Weyl chamber $W \subset C(\mathcal{I})$. Let $\mathcal{I}_1 = \mathcal{I} \smallsetminus \{\chi'\}$. Taking a generic $n \in W$, from Proposition 15.2(c) it follows for every $\lambda_j \in \mathcal{I}_1$ that

$$\gamma_n(\lambda_j \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \gamma_n(\lambda_j \mid \mathcal{F} \vee \eta \mid \mathcal{I}_1).$$

Proceeding recursively, we define $\mathcal{I} \supset \mathcal{I}_1 \supset \cdots \supset \mathcal{I}_r := \chi$ with

$$\gamma_n(\lambda_j \mid \mathcal{F} \vee \eta \mid \mathcal{I}) = \gamma_n(\lambda_j \mid \mathcal{F} \vee \eta \mid \mathcal{I}_j)$$

for each $1 \leq j \leq r$. □

16. Proof of Theorems 13.5 and 13.6

We retain all notation from Section 13.2. In particular $\hat{\alpha}$ is a measurable factor of α induced by ψ and $\mathcal{A}^\psi$ is the α-invariant measurable partition on (M, μ) induced by the factor map ψ.

16.1. **Key Lemma.** Consider any non-zero exponent $\hat{\chi} \in \hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$. Note that we not yet show the equivalence class $\hat{\chi}$ is an element of $\hat{\mathcal{L}}^{\alpha}(\mu)$. Take $H = C(\hat{\chi})$ to be the Lyapunov half-space associated with $\hat{\chi}$.

Considering all non-zero $\hat{\chi}' \in \mathcal{L}^{\hat{\alpha}}(\nu)$ and $\chi' \in \mathcal{L}^{\alpha}(\mu)$ as Lyapunov functionals on $\mathbb{R}^d$, consider *joint Weyl chambers* as connected subsets of the complement of all Lyapunov hyperplanes of all non-zero $\hat{\chi}' \in \mathcal{L}^{\hat{\alpha}}(\nu)$ and $\chi' \in \mathcal{L}^{\alpha}(\mu)$. Then H is saturated by joint Weyl chambers and we may take a finite set $S \subset H \cap \mathbb{Z}^d$ that is sufficient in H (where sufficiency is relative to the collection of joint Weyl chambers).

Take $\hat{\eta}$ a measurable partition of (N, ν) as in Proposition 14.8 that is subordinate to $\mathscr{W}^{\hat{\chi}}$ with $\hat{\alpha}(n)(\hat{\eta}) \leq \hat{\eta}$ for all $n \in S$. Let $\eta = \psi^{-1}(\hat{\eta})$.

The key observation in the proof of Theorems 13.5 and 13.6 is that for $n \in S \subset H$, every coarse exponent $\chi \in \hat{\mathcal{L}}^{\alpha}(\mu)$ with $\chi \neq \hat{\chi}$ only contributes fiber-entropy to $h_\mu(\alpha(n) \mid$

η). Let

$$\mathcal{U}^\alpha_\mu(n) := \{\chi' \in \hat{\mathcal{L}}^\alpha(\mu) : \chi'(n) > 0\}.$$

Lemma 16.1. *For $n \in S \subset H$ and $\chi \in \hat{\mathcal{L}}^\alpha(\mu)$ with $\chi \neq \hat{\chi}$ and $\chi(n) > 0$ we have*

$$\gamma_n(\lambda_j \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) = \gamma_n(\lambda_j \mid \mathcal{A}^\psi \mid \chi)$$

for all $\lambda_j \in \chi$.

Proof. First note that for any $n \in S$ with $\chi(n) > 0$, from Claim 15.1 we have

$$\gamma_n(\lambda_j \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) \geq \gamma_n(\lambda_j \mid \eta \mid \chi) \geq \gamma_n(\lambda_j \mid \mathcal{A}^\psi \mid \chi) \tag{34}$$

for all $\lambda_j \in \chi$.

To prove the reverse inequality, note that as $\chi \neq \hat{\chi}$ the Lyapunov hyperplane associated to χ intersects the interior of H. In particular, there are joint Weyl chambers $W_1 \subset H$ and $W_2 \subset H$ that are adjacent through χ and $n_i \in W_i \cap S \subset H$ so that (c) of Definition 14.3 holds with $\chi(n_1) > 0$ and $\chi(n_2) < 0$.

Let $W \subset C(\chi) \cap H$ be the joint Weyl chamber containing the n in the lemma. As S is sufficient in $C(\chi) \cap H$ there is a sequence of subsequently adjacent joint Weyl chambers $W = W^1, W^1, \dots, W^{\ell+1} = W_1$ in $C(\chi) \cap H$ and a sequence of coarse exponents $\chi'_1, \chi'_2, \dots, \chi'_\ell$ in $\hat{\mathcal{L}}^\alpha(\mu)$ with $\chi'_j \neq \chi$ for $1 \leq j \leq \ell$ such that each pair W^j, W^{j+1} is adjacent through χ'_j. Note also that since each $W^j \subset H$, we have $\chi'_j \neq \hat{\chi}$ for $1 \leq j \leq \ell$. Let $\mathcal{I} = \mathcal{U}^\alpha_\mu(n) \setminus \{\chi'_1, \chi'_2, \dots, \chi'_\ell\}$. Then also $\mathcal{I} = \mathcal{U}^\alpha_\mu(n_1) \setminus \{\chi'_1, \chi'_2, \dots, \chi'_\ell\}$. From Proposition 15.2 we have for $\lambda_j \in \chi$

(1) $\gamma_n(\lambda_j \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) = \gamma_{n_1}(\lambda_j \mid \eta \mid \mathcal{I})$;
(2) $\gamma_{n_1}(\lambda_j \mid \eta \mid \mathcal{U}^\alpha_\mu(n_1)) = \gamma_n(\lambda_j \mid \eta \mid \mathcal{I})$;
(3) $\gamma_n(\lambda_j \mid \eta \mid \mathcal{I}) = \gamma_{n_1}(\lambda_j \mid \eta \mid \mathcal{I})$;
(4) $\gamma_n(\lambda_j \mid \mathcal{A}^\psi \mid \chi) = \gamma_{n_1}(\lambda_j \mid \mathcal{A}^\psi \mid \chi)$.

To prove the lemma, it is thus sufficient to show for $\lambda_i \in \chi$ that

$$\gamma_{n_1}(\lambda_j \mid \eta \mid \mathcal{U}^\alpha_\mu(n_1)) = \gamma_{n_1}(\lambda_j \mid \mathcal{A}^\psi \mid \chi). \tag{35}$$

From Theorem 7.7, for any $n \in S$ we have

$$h_\mu(\alpha(n) \mid \eta) = \sum_{\lambda_i \in \mathcal{U}^\alpha_\mu(n)} \lambda_i(n) \gamma_n(\lambda_i \mid \eta \mid \mathcal{U}^\alpha_\mu(n)).$$

Moreover, from the Abramov–Rohlin formula (18), Theorem 7.7, and Theorem 13.1 (applied to the trivial foliation and α-invariant partition $\mathcal{A}^\psi$) we have for any $n \in \mathbb{Z}^d$ that

$$\begin{aligned} h_\mu(\alpha(n) \mid \eta) &= h_\nu(\hat{\alpha}(n) \mid \hat{\eta}) + h_\mu(\alpha(n) \mid \mathcal{A}^\psi) \\ &= h_\nu(\hat{\alpha}(n) \mid \hat{\eta}) + \sum_{\chi \in \mathcal{U}^\alpha_\mu(n)} h_\mu(\alpha(n) \mid \mathscr{W}^\chi \vee \mathcal{A}^\psi). \\ &= h_\nu(\hat{\alpha}(n) \mid \hat{\eta}) + \sum_{\chi' \in \mathcal{U}^\alpha_\mu(n)} \sum_{\lambda_i \in \chi'} \lambda_i(n) \gamma_n(\lambda_i \mid \mathcal{A}^\psi \mid \chi'). \end{aligned}$$

whence

$$h_\nu(\hat{\alpha}(n) \mid \hat{\eta}) = \sum_{\lambda_i(n) \in \mathcal{U}^\alpha_\mu(n)} \lambda_i(n) \gamma_n(\lambda_i \mid \eta \mid \mathcal{U}(n)) - \sum_{\chi' \in \mathcal{U}^\alpha_\mu(n)} \sum_{\lambda_i \in \chi'} \lambda_i(n) \gamma_n(\lambda_i \mid \mathcal{A}^\psi \mid \chi').$$

It follows from Proposition 14.8 that on $S \subset H$, that the map

$$n \mapsto h_\nu(\hat{\alpha}(n) \mid \hat{\eta})$$

coincides with a linear function $L\colon \mathbb{R}^d \to \mathbb{R}$. From Propositions 14.8 and 15.2 it follows that for all $n \in W_1 \cap S$ and $m \in W_2 \cap S$

$$L(n) = \sum_{\lambda_i \in \mathcal{U}_\mu^\alpha(n_1)} \lambda_i(n)\gamma_{n_1}(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n_1)) - \sum_{\chi' \in \mathcal{U}_\mu^\alpha(n_1)} h_\mu(\alpha(n) \mid \mathscr{W}^{\chi'} \vee \mathcal{A}^\psi) \quad (36)$$

$$L(m) = \sum_{\lambda_i \in \mathcal{I}^*} \lambda_i(m)\gamma_{n_2}(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n_2)) - \sum_{\chi' \in \mathcal{I}*} h_\mu(\alpha(m) \mid \mathscr{W}^{\chi'} \vee \mathcal{A}^\psi). \quad (37)$$

where $\mathcal{I}^* = (\mathcal{U}_\mu^\alpha(n_1) \smallsetminus \chi) \cup \{-\chi\}$ or $\mathcal{I}^* = \mathcal{U}_\mu^\alpha(n_1) \smallsetminus \chi$ depending, respectively, on whether or not $-\chi$ is a coarse exponent in $\hat{\mathcal{L}}^\alpha(\mu)$.

From Proposition 15.2, for $\chi' \neq \chi$ and $\chi' \neq -\chi$ and $\lambda_i \in \chi'$ we have equalities

$$\gamma_{n_1}(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n_1)) = \gamma_{n_2}(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n_2))$$
$$\gamma_{n_1}(\lambda_i \mid \mathcal{A}^\psi \mid \chi') = \gamma_{n_2}(\lambda_i \mid \mathcal{A}^\psi \mid \chi').$$

Let L_1 be the function

$$L_1(n) = \sum_{\lambda_i \in \chi} \lambda_i(n)\gamma_{n_1}(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n_1)) - \sum_{\lambda_i(n) \in \chi} \lambda_i(n)\gamma_{n_1}(\lambda_i \mid \mathcal{A}^\psi \mid \chi)$$

and (if $-\chi$ is a coarse Lyapunov exponent) let

$$L_2(n) = \sum_{\lambda_i \in -\chi} \lambda_i(n)\gamma_{n_2}(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n_2)) - \sum_{\lambda_i(n) \in -\chi} \lambda_i(n)\gamma_{n_2}(\lambda_i \mid \mathcal{A}^\psi \mid -\chi).$$

Comparing righthand sides of (36) and (37) and canceling common linear terms it follows that either

$$L_1 = 0 \qquad \text{or} \qquad L_1 = L_2.$$

From (34) we have $L_1(n) \geq 0$ and $L_2(n) \leq 0$ for $n \in W_1$ whence either case above implies

$$\sum_{\lambda_i \in \overline{\chi}} \lambda_i(n_1)\left(\gamma_{n_1}(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n_1)) - \gamma_{n_1}(\lambda_i \mid \mathcal{A}^\psi \mid \chi)\right) = 0.$$

(35) then follows from (34). □

From Lemma 16.1 we obtain Theorem 13.5.

Proof of Theorem 13.5. We retain all notations from above. In particular, we take $\hat{\chi} \in \hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$ with

$$h_\nu(\hat{\alpha}(n) \mid \hat{\chi}) > 0$$

for some n with $\hat{\chi}(n) > 0$ and fix $\hat{\eta}$ as above. Suppose that every coarse exponent $\chi \in \hat{\mathcal{L}}^\alpha(\mu)$ is distinct from $\hat{\chi}$. Then, by the Abramov-Rohlin formula (19), Theorem 7.7, and Lemma 16.1 we obtain a contradiction as for any $n \in S \subset H$ we have

$$\begin{aligned} h_\nu(\hat{\alpha}(n) \mid \mathscr{W}^{\hat{\chi}}) &= h_\nu(\hat{\alpha}(n) \mid \hat{\eta}) \\ &= h_\mu(\alpha(n) \mid \eta) - h_\mu(\alpha(n) \mid \mathcal{A}^\psi) \\ &= \sum_{\lambda_i \in \mathcal{U}_\mu^\alpha(n)} \lambda_i(n)\gamma_n(\lambda_i \mid \eta \mid \mathcal{U}_\mu^\alpha(n)) - \sum_{\chi \in \hat{\mathcal{L}}^\alpha(\mu): \chi(n) > 0} \sum_{\lambda_i \in \chi} \lambda_i(n)\gamma_n(\lambda_i \mid \mathcal{A}^\psi \mid \chi) \\ &= 0. \end{aligned}$$

□

Having established Theorem 13.5, given any coarse exponent $\hat{\chi} \in \hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$ contributing entropy to the factor system $\hat{\alpha}$ it follows that $\hat{\chi} \in \hat{\mathcal{L}}^\alpha(\mu)$. That is, if

$$\nu(\hat{\alpha}(n) \mid \mathscr{W}^\chi) > 0$$

for some n, then $\hat{\chi}$ is also a coarse exponent for the action of α on (M, μ). With $\hat{\eta}$, η, and $S \subset H = C(\hat{\chi})$ as in Lemma 16.1, we obtain the following.

Corollary 16.2. *For $n \in S$ we have*

$$h_\mu(\alpha(n) \mid \mathscr{W}^{\hat{\chi}} \vee \eta) = h_\nu(\hat{\alpha}(n), \hat{\eta}) + h_\mu(\alpha(n) \mid \mathcal{A}^\psi \vee \mathscr{W}^{\hat{\chi}})$$

Proof. Note from Corollary 15.3 that for $n \in S$ and $\chi' \subset \mathcal{U}^\alpha_\mu(n)$, for $\lambda_i \in \chi'$

$$\gamma_n(\lambda_i \mid \mathcal{A}^\psi \mid \chi') = \gamma_n(\lambda_i \mid \mathcal{A}^\psi \mid \mathcal{U}^\alpha_\mu(n))$$

and is defined independent of n. Let $\gamma(\lambda_i \mid \mathcal{A}^\psi)$ denote this constant.

From Lemma 16.1 and Theorem 7.7 we have

$$\begin{aligned} h_\mu(\alpha(n) \mid \eta) &= \sum_{\lambda_i(n) \in \mathcal{U}^\alpha_\mu(n)} \lambda_i(n) \gamma_n(\lambda_i \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) \\ &= \sum_{\lambda_i(n) \in \mathcal{U}^\alpha_\mu(n)} \lambda_i(n) \gamma(\lambda_i \mid \mathcal{A}^\psi) + \sum_{\lambda_i(n) \in \hat{\chi}} \lambda_i(n) \left(\gamma_n(\lambda_i \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) - \gamma(\lambda_i \mid \mathcal{A}^\psi)\right) \\ &= h_\mu(\alpha(n) \mid \mathcal{A}^\psi) + \sum_{\lambda_i(n) \in \hat{\chi}} \lambda_i(n) \left(\gamma_n(\lambda_i \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) - \gamma(\lambda_i \mid \mathcal{A}^\psi)\right). \end{aligned}$$

whence

$$\begin{aligned} h_\nu(\hat{\alpha}(n), \hat{\eta}) &= h_\mu(\alpha(n) \mid \eta) - h_\mu(\alpha(n) \mid \mathcal{A}^\psi) \\ &= \sum_{\lambda_i(n) \in \hat{\chi}} \lambda_i(n) \left(\gamma_n(\lambda_i \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) - \gamma(\lambda_i \mid \mathcal{A}^\psi)\right). \end{aligned}$$

From Proposition 15.2 (taking a sequence of points in S meeting every Weyl chamber of $\hat{\mathcal{L}}^\alpha(\mu)$ in $H = C(\hat{\chi})$) we have for $\lambda_i \in \hat{\chi}$ that

$$\gamma_n(\lambda_i \mid \eta \mid \mathcal{U}^\alpha_\mu(n)) = \gamma_n(\lambda_i \mid \eta \mid \hat{\chi}).$$

Thus

$$\begin{aligned} h_\nu(\hat{\alpha}(n), \hat{\eta}) &= \sum_{\lambda_i(n) \in \hat{\chi}} \lambda_i(n) \left(\gamma_n(\lambda_i \mid \eta \mid \hat{\chi}) - \gamma(\lambda_i \mid \mathcal{A}^\psi)\right) \\ &= h_\mu(\alpha(n) \mid \mathscr{W}^{\hat{\chi}} \vee \eta) - h_\mu(\alpha(n) \mid \mathcal{A}^\psi \vee \mathscr{W}^{\hat{\chi}}). \end{aligned}$$ □

16.2. **Proof of Theorem 13.6.** We proceed with the proof of the theorem.

Proof of Theorem 13.6. Consider any fixed, non-zero $\chi \in \hat{\mathcal{L}}^\alpha(\mu)$. Fix $n \in \mathbb{Z}^d$ with $\chi(n) > 0$ and let W be the Weyl chamber of $\hat{\mathcal{L}}^\alpha(\mu)$ containing n. Fix a finite set S that is sufficient in W.

Choose an enumeration of all coarse exponents $\chi_j \subset \mathcal{U}^\alpha_\mu(n)$ with $\chi = \chi_0$. For each $\chi_j \in \hat{\mathcal{L}}^\alpha(\mu)$ let $\hat{\chi}_j = \chi_j$ if χ_j is a coarse Lyapunov exponent in $\hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$; otherwise, let $\hat{\chi}_j$ denote the 0 functional. For each $\chi_j \subset \mathcal{U}^\alpha_\mu(n)$ with $\hat{\chi}_j = \chi_j$ let $\hat{\eta}_j$ be a measurable partition of (N, ν) as in Proposition 14.8 that is subordinate to $\mathscr{W}^{\hat{\chi}_j}$ with $\hat{\alpha}(m)(\hat{\eta}_j) \leq \hat{\eta}_j$ for all $m \in W \cap S$. If $\hat{\chi}_j$ is not a coarse exponent in $\hat{\mathcal{L}}^{\hat{\alpha}}(\nu)$ take $\hat{\eta}_j$ to be the point partition. Let $\eta_j = \psi^{-1}(\hat{\eta}_j)$.

From Theorems 13.1 and 13.5 we have for $m \in W \cap S$

(1) $h_\nu(\hat{\alpha}(m)) = \sum_{\chi_j \in \mathcal{U}^\alpha_\mu(m)} h_\nu(\hat{\alpha}(m), \hat{\eta}_j)$;

(2) $h_\mu(\alpha(m) \mid \mathcal{A}^\psi) = \sum_{\chi_j \in \mathcal{U}^\alpha_\mu(m)} h_\mu(\alpha(m) \mid \mathcal{A}^\psi \vee \mathscr{W}^{\chi_j})$;

(3) $h_\mu(\alpha(m)) = \sum_{\chi_j \in \mathcal{U}_\mu^\alpha(m)} h_\mu(\alpha(m) \mid \mathscr{W}^{\chi_j})$.

Combined with Corollary 16.2, it follows that for $m \in W \cap S$,

$$\begin{aligned}\sum_{\chi_j \in \mathcal{U}_\mu^\alpha(m)} h_\mu(\alpha(m) \mid \mathscr{W}^{\chi_j} \vee \eta_j) &= h_\nu(\hat{\alpha}(m)) + h_\mu(\alpha(m) \mid \mathcal{A}^\psi) \\ &= \sum_{\chi_j \in \mathcal{U}_\mu^\alpha(m)} h_\mu(\alpha(m) \mid \mathscr{W}^{\chi_j}).\end{aligned} \tag{38}$$

From Claim 8.6, for each j we have

$$h_\mu(\alpha(m) \mid \mathscr{W}^{\chi_j} \vee \eta_j) \leq h_\mu(\alpha(m) \mid \mathscr{W}^{\chi_j})$$

for every $m \in W \cap S$. Combined with (38) it follows for each j and $m \in W \cap S$ that

$$h_\mu(\alpha(m) \mid \mathscr{W}^{\chi_j} \vee \eta_j) = h_\mu(\alpha(m) \mid \mathscr{W}^{\chi_j})$$

whence the result follows for $m \in W \cap S$ from Corollary 16.2. The result then follows for all $m \in W$ by linearity. □

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