Arbitrarily slow decay in the Möbius disjointness conjecture

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Abstract

Sarnak's Möbius disjointness conjecture asserts that for any zero entropy dynamical system (X,T), $\frac{1}{N}\sum_{n=1}^{N} f(T^n x)\mu(n) = o(1)$ for every $f \in \mathcal{C}(X)$ and every $x \in X$. We construct examples showing that this $o(1)$ can go to zero arbitrarily slowly. In fact, our methods yield a more general result, where in lieu of $\mu(n)$ one can put any bounded sequence a_n such that the Cesàro mean of the corresponding sequence of absolute values does not tend to zero. Moreover, in our construction the choice of x depends on the sequence a_n but (X,T) does not.

1 Introduction

A topological dynamical system is a pair (X,T) where X is compact metric space and $T \in \mathcal{C}(X)$. If the system (X,T) has zero topological entropy, then Sarnak's Möbius disjointness conjecture [15, Main Conjecture] predicts that

$$\frac{1}{N}\sum_{n=1}^{N} \mu(n) f(T^n x) = o(1), \quad \text{for every } f \in \mathcal{C}(X) \text{ and every } x \in X. \tag{1}$$

Many special cases of Sarnak's Conjecture have been established: A very partial list of examples consists of [2, 5, 7, 9]. We refer to the surveys of Ferenczi, Kułaga-Przymus, and Lemańczyk [6] and of Kułaga-Przymus and Lemańczyk [11] for excellent expositions on the subject, and many more references.

The goal of this paper is to study the rate of decay in Sarnak's conjecture. That is, to study the nature of the $o(1)$ as in (1). We will show that there are systems for which this $o(1)$ decays to zero arbitrarily slowly. Nevertheless, all the examples we construct to this end satisfy Sarnak's conjecture. Here is our main result:

Theorem 1.1. *For every decreasing and strictly positive sequence $\tau(n) \to 0$ there is a dynamical system (X,T) with zero topological entropy that satisfies:*

1. *There exist $x \in X$ and $f \in \mathcal{C}(X)$ such that:*

$$\limsup_{N\to\infty} \frac{\frac{1}{N}\sum_{n=1}^{N} f(T^n x)\mu(n)}{\tau(n)} > 0.$$

2. *The system (X,T) satisfies Sarnak's conjecture (1).*

Several remarks are in order. First, Sarnak [14, the remark following Main Conjecture] remarks that rates are not required in the conjecture, and this is formally justified by Theorem 1.1. Secondly, it is natural to ask if Theorem 1.1 may be upgraded by finding a zero entropy dynamical system (X,T) and $f \in \mathcal{C}(X)$ such that for every rate function τ we can find $x \in X$ that satisfies part (1) of Theorem 1.1. Doing so is as hard as solving the full Möbius disjointness conjecture: Indeed, by [4, Corollary 10], if the conjecture is true then for every zero entropy system (X,T) and $f \in \mathcal{C}(X)$, (1) holds uniformly in $x \in X$. This cannot hold concurrently with the aforementioned upgraded version of Theorem 1.1. In other words, Theorem 1.1 is conjecturally optimal. Next, we remark that in many cases (possibly in all cases), it is known [16] that a sufficiently fast rate in Sarnak's conjecture implies that the system (X,T) satisfies a prime number Theorem (PNT) in the sense discussed in [6, Section 11.2]. Thus, recent examples [10, 8] of zero entropy systems failing to satisfy a PNT can be viewed as evidences towards Theorem 1.1. We also mention some recent interesting examples constructed by Lian and Shi [12] that, while not directly related to Theorem 1.1, are similar in spirit to our work. Finally, we remark that our construction was partially inspired by the recent work of Dolgopyat, Dong, Kanigowski, and Nándori [3], where they exhibit some new classes of zero entropy smooth systems that satisfy the Central Limit Theorem

We will derive Theorem 1.1 from a more general statement. This is the following Theorem, which forms the main technical result of this paper:

Theorem 1.2. *For every decreasing and strictly positive sequence $\tau(n) \to 0$ there is a zero entropy dynamical system (X,T) and some $f \in \mathcal{C}(X)$ that satisfy:*

1. *Every sequence $|a_n| \leq 1$ with $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |a_n| > 0$ admits some $x \in X$ such that*

$$\limsup_{N\to\infty} \frac{\frac{1}{N}\sum_{n=1}^{N} f(T^n x) a_n}{\tau(n)} > 0.$$

2. *The system (X,T) satisfies Sarnak's conjecture (1).*

In fact, we will show that any sub-sequence N_j such that

$$\lim_{j\to\infty} \frac{1}{N_j}\sum_{n=1}^{N_j} |a_n| = \theta > 0 \tag{2}$$

admits a further subsequence N_{j_k} such that for all k large enough

$$\frac{1}{N_{j_k}}\sum_{n=1}^{N_{j_k}} f(T^n x) a(n) \geq \theta \cdot \tau(N_{j_k}).$$

We emphasize that in Theorem 1.2 the system (X,T) and the function $f \in \mathcal{C}(X)$ only depend on the rate function τ, while the point $x \in X$ depends also on the sequence a_n.

The derivation of Theorem 1.1 from Theorem 1.2 is straightforward: It is well known that the Möbius function μ satisfies

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} |\mu(n)|}{N} = \frac{6}{\pi^2} > 0,$$

see e.g. [1, Corollary 1.6]. Thus, Theorem 1.2 applied with $a_n = \mu(n)$ gives Theorem 1.1.

We end this introduction with a brief explanation of our construction. We consider subshifts of $\left(\{-1,0,1\}^{\mathbb{N}} \times \{-1,0,1\}^{\mathbb{Z}}, T\right)$, where $T(y,z) = (\sigma y, \sigma^{y_1} z)$ and σ is the left shift. Given a rate function τ we first construct a certain rapidly growing sequence $q_k \to \infty$. We then construct a subshift such that its base comes from concatenating words of length $q_{k+1} - q_k$, that have non-zero entries at distance at least q_k from each other. Our space X is a product of 4 spaces constructed this way, together with a finite set $\{0,1,2,3\}$. The function f is taken to be

$$f((y^{(0)}, z^{(0)}),\, (y^{(1)}, z^{(1)}),\, (y^{(2)}, z^{(2)}),\, (y^{(3)}, z^{(3)}),\, i) = z_0^{(i)}.$$

Given a_n as in Theorem 1.2 part (1), our construction of the point $x \in X$ relies on the following observation: Assuming $a_n \in \mathbb{R}$ (see the beginning of Section 2.3 on why this is allowed), let $\gamma_n := \text{sign}(a_n)$, and let N_j, θ be as in (2). For every $q, M \gg 1$ one may show that

$$\max_{c,d \in [0,q] \cap \mathbb{Z}} \left\{ \frac{1}{qM} \sum_{b=c}^{q-1+c} \sum_{n=1}^{M} \gamma(qn+c) \cdot a(qn+b),\ \frac{-1}{qM} \sum_{b=d}^{q-1+d} \sum_{n=1}^{M} \gamma(qn+d+1) \cdot a(qn+b) \right\} \geq \frac{\theta}{4}.$$

Here we pick $k = k(j)$ in some convenient way, $q = q_k$ and $M \approx \frac{N_j}{q_k}$. We then construct our point x via working in one of the subshifts in our space - the exact choice depends on certain technical issues coming from the relation between N_j and q_k. To set up x, we carefully concatenate pieces of arithmetic progressions in γ or $-\gamma$ in the fiber (using the equation above), with the base living in the corresponding shift space and behaving nicely along the observable f. This will allow us to find a subsequence of N_j where the linear correlations as in Theorem 1.2 part (1) are well approximated by the average giving the max in the equation above. Thus, with some more work, we bound these correlations from below by $\tau(N_j) \cdot \theta$.

Finally, to derive part (2) of Theorem 1.2, we apply the Matomäki-Radziwiłł bound [13] on averages of multiplicative functions along short intervals. To do this, we exploit some strong periodic behaviour that exists in the systems we construct.

2 Proof of Theorem 1.2 Part (1)

2.1 Preliminaries

Let (X, T) be a dynamical system, where we recall that X is a compact metric space and $T \in \mathcal{C}(X)$. We denote the metric on X by d_X. Let us recall Bowen-Dinaburg definition of topological entropy (as in e.g. [17]): For every $n \in \mathbb{N}$ we define a metric on X via

$$d_n(x, y) = \max\{d_X(T^i(x), T^i(y)) : 0 \leq i < n\}.$$

A Bowen ball $B_n(x, \epsilon)$ of depth n centred at $x \in X$ of radius $\epsilon > 0$ is the corresponding (open) ball in the metric d_n,

$$B_n(x, \epsilon) = \{y \in X : d_n(x, y) < \epsilon\}.$$

For any set $E \subseteq X$, let $N(E, n, \epsilon)$ denote the minimal number of Bowen balls of depth n and radius ϵ needed to cover E. The topological entropy of (X, T) is then defined as

$$h(T) := \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \frac{\log N(X, n, \epsilon)}{n} \right).$$

Next, let $\sigma : \{-1, 0, 1\}^{\mathbb{Z}} \to \{-1, 0, 1\}^{\mathbb{Z}}$ denote the left shift. On $\{-1, 0, 1\}^{\mathbb{Z}}$ and $\{-1, 0, 1\}^{\mathbb{N}}$ we define the metric

$$d(x, y) = 3^{-\min\{|n| :\, x_n \neq y_n\}}.$$

Also, for every $x \in \{-1, 0, 1\}^{\mathbb{N}}$ and $k > l \in \mathbb{N}$ let $x|_l^k \in \{-1, 0, 1\}^{k-l}$ be the word

$$x|_l^k := (x_l, x_{l+1}, ..., x_k),$$

and we use similar notation in the space $\{-1, 0, 1\}^{\mathbb{Z}}$ as well. Next, let

$$Z := \{-1, 0, 1\}^{\mathbb{N}} \times \{-1, 0, 1\}^{\mathbb{Z}}$$

and endow Z with the sup-metric on both its coordinates. Note that open balls in this metric are also closed, and thus for every $n \in \mathbb{N}$, $x \in X$ and $\epsilon > 0$ the Bowen ball $B_n(x, \epsilon)$ is closed. Also, we denote by Π_i, $i = 1, 2$, the coordinate projections in Z. Finally, we define the skew-product $T : Z \to Z$ via

$$T(y, z) = (\sigma(y), \sigma^{y_1}(z)).$$

We say that $X \subseteq Z$ is a subshift if it is closed and T-invariant.

We will require the following Lemma:

Lemma 2.1. *The system (Z, T) satisfies that for every $n \in \mathbb{N}$, $\epsilon > 0$, and $x = (y, z) \in Z$,*

1. *We have*
$$T^n(y,\, z) = \left(\sigma^n y,\, \sigma^{\sum_{i=1}^n y_i} z\right).$$

2. *Let $m = m(n, y) = \min\{\min_{1 \le k \le n} \sum_{i=1}^k y_i,\, 0\}$ and $M := M(n, y) = \max\{\max_{1 \le k \le n} \sum_{i=1}^k y_i,\, 0\}$. Then for any $l \in \mathbb{N}$ the Bowen ball $d_n(x, 3^{-l})$ equals*
$$\left\{(a, b) \in Z :\, a|_1^{l+n} = y|_1^{l+n},\, b|_{-l+m}^{l+M} = z|_{-l+m}^{l+M}\right\}.$$

3. *For any set $E \subseteq Z$*
$$N(E, n, \epsilon) = N(cl(E), n, \epsilon),$$
where $cl(E)$ is the closure of the set E.

Proof. Part (1) follows immediately from the definition of the map T. Part (2) follows from part (1). Finally, part (3) is an immediate consequence of the fact that in $(Z,\, T)$ Bowen balls are closed. □

2.2 Construction of some zero entropy systems

Fix a sequence $\tau(n) \to 0$ as in Theorem 1.2. We begin by constructing a rapidly growing sequence $q_k \to \infty$ (that depends on τ) such that for every $k \in \mathbb{N}$ we have:

1. $q_{k+1} > q_k^4 + 3q_k$.
2. $\tau(\frac{q_{k+1}}{3}) < \frac{1}{16 q_k}$.

We now use q_k to define four sequences:

$$q_k^{(0)} := q_{2k},\, q_k^{(1)} = q_{2k+1},\, q_k^{(2)} := q_k^{(0)} - 1,\, q_k^{(3)} := q_k^{(1)} - 1.$$

Notice that property (1) above also holds for $q_k^{(i)}$ for every $i \in \{0, 1, 2, 3\}$. In particular,

$$\lim_{k \to \infty} \frac{q_{k+1}^{(i)}}{q_k^{(i)}} = \infty, \text{ for every } i \in \{0, 1, 2, 3\}.$$

Next, for every $i \in \{0, 1, 2, 3\}$ and every k let

$$A_k^{(i)} := \{j \cdot q_k^{(i)} :\, j \in \mathbb{Z},\, q_k^{(i)} \le j \cdot q_k^{(i)} \le q_{k+1}^{(i)}\}.$$

For every $i \in \{0, 1, 2, 3\}$ and every $k \in \mathbb{N}$ we construct elements $s_k^{(i)} \in \{-1, 0, 1\}^{\mathbb{N}}$ such that:

1. $s_k^{(i)}(n) = 0$ for every integer $n \notin A_k^{(i)}$.

2. For every $j \cdot q_k^{(i)} \in A_k^{(i)}$,

$$s_k^{(i)}(j \cdot q_k^{(i)}) = 1 \text{ if } j \leq \left\lfloor \frac{q_{k+1}^{(i)}}{3q_k^{(i)}} \right\rfloor,$$

and

$$s_k^{(i)}(j \cdot q_k^{(i)}) = -1 \text{ if } \left\lfloor \frac{q_{k+1}^{(i)}}{3q_k^{(i)}} \right\rfloor < j \leq 2 \left\lfloor \frac{q_{k+1}^{(i)}}{3q_k^{(i)}} \right\rfloor.$$

Next, for every element $x \in \{-1,0,1\}^{\mathbb{N}}$ and $p \in \mathbb{N}_0$ we define $\sigma^{-p}x \in \{-1,0,1\}^{\mathbb{N}}$ as $\sigma^{-p}x = x$ if $p = 0$, and otherwise

$$\left(\sigma^{-p}x\right)|_1^p = (0, ..., 0), \text{ and for all } n > p, \ \sigma^{-p}x(n) = x(n-p).$$

The following Lemma is an immediate consequence of our construction.

Lemma 2.2. *For every $i \in \{0,1,2,3\}$, $k \in \mathbb{N}$, and $p = 0, ..., q_k^{(i)}$ we have*

$$\sum_{j \in [q_k^{(i)}, q_{k+1}^{(i)}) \cap \mathbb{Z}} \left(\sigma^{-p} s_k^{(i)}\right)(j) = 0.$$

Proof. This follows since by our construction

$$\left|\left\{j \cdot q_k^{(i)} \in A_k^{(i)} : s_k^{(i)}(j \cdot q_k^{(i)}) = 1\right\}\right| = \left|\left\{j \cdot q_k^{(i)} \in A_k^{(i)} : s_k^{(i)}(j \cdot q_k^{(i)}) = -1\right\}\right|.$$

□

Next, for every $i \in \{0,1,2,3\}$ and $k \in \mathbb{N}$ define the truncations

$$R_k^{(i)} = \left\{\left(\sigma^{-p}s_k^{(i)}\right)|_{q_k^{(i)}}^{q_{k+1}^{(i)}-1} : p = 0, ..., q_k^{(i)}\right\} \subseteq \{-1,0,1\}^{q_{k+1}^{(i)} - q_k^{(i)}}.$$

We now define the space $P^{(i)}$ of all infinite sequences that have, for every k, some word from $R_k^{(i)}$ between their $q_k^{(i)}$ and $q_{k+1}^{(i)} - 1$ digits. Formally,

$$P^{(i)} = \{x \in \{-1,0,1\}^{\mathbb{N}} : x|_{q_k^{(i)}}^{q_{k+1}^{(i)}-1} \in R_k^{(i)}, \text{ and } x|_1^{q_1^{(i)}-1} = (0, ..., 0)\}.$$

The following Lemma is an immediate consequence of Lemma 2.2:

Lemma 2.3. *For every $i \in \{0,1,2,3\}$, $k \in \mathbb{N}$, and $y \in P^{(i)}$,*

$$\sum_{j=1}^{q_k^{(i)}-1} y(j) = 0$$

Finally, for every $i \in \{0,1,2,3\}$ we define the subshift of (Z, T)

$$X_i = \mathrm{cl}\left(\bigcup_{n \in \mathbb{N}_0} T^n\left(P^{(i)} \times \{-1,0,1\}^{\mathbb{Z}}\right)\right).$$

Claim 2.4. *For every $i \in \{0, 1, 2, 3\}$ we have $h(X_i, T) = 0$.*

Proof. Fix $n, u \in \mathbb{N}$. We count how many Bowen balls of radius $\frac{1}{3^u}$ and depth n are needed to cover X_i. Recall that we denote this quantity by $N(X_i, n, \frac{1}{3^u})$. By Lemma 2.1 part (3), this is the same number as

$$N\left(\bigcup_{l\in\mathbb{N}_0} T^l\left(P^{(i)} \times \{-1,0,1\}^{\mathbb{Z}}\right),\, n,\, \frac{1}{3^u}\right).$$

So, we work with the latter space (i.e. without taking the closure).

Let $k = k(n+u, i)$ be such that

$$q_k^{(i)} \le n + u < q_{k+1}^{(i)}. \tag{3}$$

Our first observation is that we can write

$$\bigcup_{l\in\mathbb{N}_0} T^l\left(P^{(i)} \times \{-1,0,1\}^{\mathbb{Z}}\right) = A_1 \bigcup A_2 \bigcup A_3.$$

To define the sets A_i we first note that every $x \in \bigcup_{l\in\mathbb{N}_0} T^l\left(P^{(i)} \times \{-1,0,1\}^{\mathbb{Z}}\right)$ admits some $l \in \mathbb{N}_0$ and $\tilde{x} \in P^{(i)} \times \{-1,0,1\}^{\mathbb{Z}}$ such that $x = T^l\tilde{x}$. We denote by $p = p(x) \in \mathbb{N}$ the unique integer such that $q_{k-1}^{(i)} + l \in [q_p^{(i)},\, q_{p+1}^{(i)})$. Note that $p \ge k-1$. Then

$$A_1 = \{x : p(x) \ge k+1\},\ A_2 = \{x : p(x) = k\},\ A_3 = \{x : p(x) = k-1\}.$$

Thus, we bound the covering numbers for $A_1 - A_3$ separately. Before doing so, we notice that for any $x \in A_j$ for $j = 1, 2, 3$ there are at most $3^{q_{k-1}^{(i)}}$ possibilities for the first $q_{k-1}^{(i)}$ digits of $\Pi_1(x)$.

1. Covering A_1: For any $x \in A_1$ the word $(\Pi_1 x)\,|_{q_{k-1}^{(i)}}^{n+u}$ always consists of zeros separated by 1 or -1, and in this case the non-zero entries appear at distance at least $q_{k+1}^{(i)} > n+u$ from each other. Since there can be only one non-zero entry, there are at most $2(n+u)$ options for the configuration of this word. So, with the notations of Lemma 2.1 part (2), we see that

$$|m|, M \le q_{k-1}^{(i)} + 1.$$

   Thus, taking into account also the first $q_{k-1}^{(i)}$ digits, and via Lemma 2.1 part (2), the number of Bowen balls we need here is at most

$$\left(3^{q_{k-1}^{(i)}} \times 2(n+u)\right) \times \left(3^{u+q_{k-1}^{(i)}+1}\right)^2.$$

2. Covering A_2: The word $(\Pi_1 x)\,|_{q_{k-1}^{(i)}}^{n+u}$ consists of zeros separated by 1 or -1, and in this case the first non-zero entries appear at distance at least $q_k^{(i)} \le n+u$ from each other. We also know that the first non-zero digit needs to appear within the first $q_k^{(i)}$ digits. Another factor that needs to be taken into consideration is the possibility that $[q_{k-1}^{(i)} + l,\, n+u+l]$ intersects $[q_{k+1}^{(i)}, \infty)$. So, with the notations of Lemma 2.1 part (2), we see that

$$|m|, M \le q_{k-1}^{(i)} + \frac{n+u}{q_k^{(i)}} + 1.$$

Taking all these factor into account, the number of Bowen balls we need here is at most

$$\left(3^{q_{k-1}^{(i)}} \times q_k^{(i)} \times 2(n+u)\right) \times \left(3^{u+q_{k-1}^{(i)}+\frac{n+u}{q_k^{(i)}}+1}\right)^2.$$

3. Covering A_3: The word $(\Pi_1 x)\,|_{q_{k-1}^{(i)}}^{n+u}$ consists of zeros separated by 1 or -1, and in this case the first non-zero entries appear at distance at least $q_{k-1}^{(i)}$ from each other. We also know that the first non-zero digit needs to appear within the first $q_{k-1}^{(i)}$ digits. Another factor that needs to be taken into consideration is the possibility that $[q_{k-1}^{(i)}+l,\, n+u+l]$ intersects $[q_k^{(i)}, \infty)$. So, with the notations of Lemma 2.1 part (2), we see that

$$|m|, M \leq q_{k-1}^{(i)} + \frac{q_k^{(i)}}{q_{k-1}^{(i)}} + \frac{n+u}{q_k^{(i)}} + 1.$$

Taking all these factor into account, the number of Bowen balls we need here is at most

$$\left(3^{q_{k-1}^{(i)}} \times q_{k-1}^{(i)} \times q_k^{(i)} \times 2(n+u)\right) \times \left(3^{u+q_{k-1}^{(i)}+\frac{q_k^{(i)}}{q_{k-1}^{(i)}}+\frac{n+u}{q_k^{(i)}}+1}\right)^2.$$

Thus, we see that

$$N(X_i, n, \frac{1}{3^u}) \leq 3 \cdot \max_{i=1,2,3} N(A_i, n, \frac{1}{3^u}) = 3 \cdot N(A_3, n, \frac{1}{3^u}),$$

which has been computed in point (3) above. So, making use of (3),

$$\begin{aligned}
\frac{\log N(X_i, n, \frac{1}{3^u})}{n} &\leq \frac{\log 3 + \log\left(3^{q_{k-1}^{(i)}} \cdot 2(n+u) \cdot q_{k-1}^{(i)} \cdot q_k^{(i)}\right) \cdot \left(3^{u+q_{k-1}^{(i)}+\frac{q_k^{(i)}}{q_{k-1}^{(i)}}+\frac{n+u}{q_k^{(i)}}+1}\right)^2}{n} \\
&\leq \frac{\log 6}{n} + \frac{q_{k-1}^{(i)} \cdot \log 3}{n} + \frac{\log(n+u)}{n} + \frac{2\log q_k^{(i)}}{n} \\
&\quad + \frac{\left(u + q_{k-1}^{(i)} + \frac{q_k^{(i)}}{q_{k-1}^{(i)}} + \frac{n+u}{q_k^{(i)}} + 1\right)\log 9}{n} \\
&\leq C_1 \cdot \left(\frac{\log q_k^{(i)}}{n} + \frac{q_{k-1}^{(i)}}{n} + \frac{\log(n+u)}{n} + \frac{q_k^{(i)}}{q_{k-1}^{(i)} \cdot (n+u)} \cdot \frac{n+u}{n} + \frac{n+u}{q_k^{(i)} \cdot n}\right) \\
&\leq C_1 \cdot \frac{n+u}{n} \cdot \left(\frac{2\log(n+u)}{n} + \frac{q_{k-1}^{(i)}}{q_k^{(i)}} + \frac{1}{q_{k-1}^{(i)}} + \frac{1}{q_k^{(i)}}\right).
\end{aligned}$$

Here C_1 is a large constant that depends variously on u and the other constants appearing in the second equation. We conclude that, fixing u,

$$\lim_{n\to\infty} \frac{\log N(X, n, \frac{1}{3^u})}{n} = 0,$$

and the Claim is proved. □

2.3 Finding correlations along arithmetic progressions

Let a_n be a sequence as in part (1) of Theorem 1.2, that is, such that $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |a_n| > 0$. By moving to either $\Re(a_n)$ or $\Im(a_n)$, we may assume a_n is a real valued sequence. We define a new sequence $\gamma_n \in \{-1,0,1\}$ via

$$\gamma_n := \operatorname{sign}(a_n).$$

In particular,

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \gamma_n \cdot a_n = \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |a_n| > 0.$$

Let $\theta := \limsup \frac{1}{N}\sum_{n=1}^{N} |a_n| > 0$, and let N_j be a sub-sequence such that

$$\lim_{j\to\infty} \frac{1}{N_j}\sum_{n=1}^{N_j} |a_n| = \theta.$$

Definition 2.5. *For every $j \in \mathbb{N}$ large enough we define $k' = k(j) \in \mathbb{N}$ and $i' = i(j) \in \{0,1\}$ as the unique integers such that:*

$$\textit{If } N_j \in [\frac{q_{k'}^{(1)}}{3}, \frac{q_{k'+1}^{(0)}}{3}) \textit{ then } i' = 0, \textit{ and}$$

$$\textit{If } N_j \in [\frac{q_{k'+1}^{(0)}}{3}, \frac{q_{k'+1}^{(1)}}{3}) \textit{ then } i' = 1.$$

We also define an integer

$$M_{k'}^{(i')} := \left[\frac{N_j}{q_{k'}^{(i')}}\right]$$

Note that by definition and the construction of the sequence q_k

$$\frac{\left(q_{k'}^{(i')}\right)^3}{3} = \frac{\left(q_{k'}^{(i')}\right)^4}{3q_{k'}^{(i')}} < M_{k'}^{(i')} \le \frac{q_{k'+1}^{(i')}}{3q_{k'}^{(i')}}. \tag{4}$$

Next, recall the definition of Z from Section 2.1 and let $g : Z \to \{-1,0,1\}$ be the function

$$g(y,z) = z_0.$$

For every $q, M \gg 1$ and r, c such that $r, c \in [0,q]$ let

$$A_{r,c}^{q,M} := \frac{1}{qM}\sum_{b=r}^{q-1+r}\sum_{n=1}^{M} \gamma(qn+c)\cdot a(qn+b).$$

Finally, we also define

$$M_{k'}^{(i'+2)} := \left[\frac{q_{k'}^{(i')} M_{k'}^{(i')}}{q_{k'}^{(i')} - 1}\right] = \left[\frac{q_{k'}^{(i')} M_{k'}^{(i')}}{q_{k'}^{(i'+2)}}\right]$$

and note that $M_{k'}^{(i'+2)} \approx M_{k'}^{(i')}$. In the following Lemma we use the construction from Section 2.2.

Lemma 2.6. *For every j and $u \in \{0,1\}$, writing $\ell = i' + 2u$, for every two integers $c, r \in [0, q_{k'}^{(\ell)}]$ let $x \in P^{(\ell)} \times \{-1,0,1\}^{\mathbb{Z}} \subseteq X_\ell$ be any element such that for every $q_{k'}^{(\ell)} \le n < q_{k'+1}^{(\ell)}$*

$$x(n) = \left(s_{k'}^{(\ell)}(n-r),\, \gamma(q_{k'}^{(\ell)} \cdot n + c)\right).$$

Then

$$\frac{1}{q_{k'}^{(\ell)} M_{k'}^{(\ell)}} \sum_{n=1}^{q_{k'}^{(\ell)} M_{k'}^{(\ell)}} g(T^n x) a(n) = A_{r,c}^{q_{k'}^{(\ell)}, M_{k'}^{(\ell)}} + O\left(\frac{q_{k'}^{(\ell)}}{M_{k'}^{(\ell)}}\right).$$

Note that by the construction of $P^{(\ell)} \times \{-1,0,1\}^{\mathbb{Z}}$ in Section 2.2, there exists an element x as in the statement of the Lemma in that space.

Proof. In this proof we suppress the ℓ, k' in our notation and simply write q, M. First, for every two integers $j \in [1,\, M]$ and $b \in [r,\, q+r-1]$,

$$\begin{aligned}
\sum_{d=1}^{qj+b} (\Pi_1 x)(d) &= \sum_{d=1}^{q-1} (\Pi_1 x)(d) + \sum_{d=q}^{qj+b-1} (\Pi_1 x)(d) \\
&= \sum_{d=q}^{qj+b-1} s_k^{(\ell)}(d-r) \\
&= \sum_{d=q-r}^{qj+b-r-1} s_k^{(\ell)}(d) = j.
\end{aligned}$$

Note the use of Lemma 2.3 in the second equality, and the use of the definition of $s_k^{(\ell)}$ together with the fact that $M \le \frac{q_{k'+1}^{(\ell)}}{3q_{k'}^{(\ell)}}$ in the last one. Therefore,

$$\begin{aligned}
\frac{1}{qM}\sum_{n=1}^{qM} g(T^n x)a(n) &= \frac{1}{qM}\sum_{n=q}^{qM} g(T^n x)a(n) + O\left(\frac{1}{M}\right) \\
&= \frac{1}{qM}\sum_{j=1}^{M}\sum_{b=r}^{q+r-1} g(T^{q\cdot j+b}x)a(q\cdot j+b) + O\left(\frac{1}{M}\right) \\
&= \frac{1}{qM}\sum_{j=1}^{M}\sum_{b=r}^{q+r-1} g\left(\sigma^{qj+b}\Pi_1 x,\, \sigma^{\sum_{d=1}^{qj+b}(\Pi_1 x)(d)}\Pi_2 x\right) a(q\cdot j+b) \\
&\quad + O\left(\frac{1}{M}\right) \\
&= \frac{1}{qM}\sum_{j=1}^{M}\sum_{b=r}^{q+r-1} g\left(\sigma^{qj+b-r} s_{k'}^{(\ell)},\, \sigma^j \Pi_2 x\right) a(q\cdot j+b) \\
&\quad + O\left(\frac{1}{M}\right) \\
&= \frac{1}{qM}\sum_{j=1}^{M}\sum_{b=r}^{q+r-1} \gamma(q\cdot j+c)\cdot a(q\cdot j+b) + O\left(\frac{q}{M}\right)
\end{aligned}$$

$$= \quad A^{q,M}_{r,c} + O\left(\frac{q}{M}\right).$$

Indeed: The first equality follows since $g(T^n x)$ and a_n are both bounded sequences, in the third equality we use Lemma 2.1 part (1), and in the fourth equality we are using the previous equation array and the definition of x. This definition along with the definition of $s^{(\ell)}_k$ justify the fifth equality. The last equality is simply the definition of $A^{q,M}_{r,c}$. □

Remark 2.7. *In the setup of Lemma 2.6, we may similarly find another $x \in P^{(\ell)} \times \{-1,0,1\}^{\mathbb{Z}}$ that satisfies the conclusion of Lemma 2.6, but for $-A^{q^{(\ell)}_{k'},M^{(\ell)}_{k'}}_{r,c}$. Indeed, this follows from the very same proof by picking $x \in P^{(\ell)} \times \{-1,0,1\}^{\mathbb{Z}}$ to be any element such that for every $q^{(\ell)}_{k'} \le n < q^{(\ell)}_{k'+1}$*

$$x(n) = \left(s^{(\ell)}_{k'}(n-r),\ -\gamma(q^{(\ell)}_{k'} \cdot n + c)\right).$$

We will also require the following Lemma:

Lemma 2.8. *For every j large enough there is either some $c \in [0, q^{(i')}_{k'})$ such that*

$$A^{q^{(i')}_{k'},M^{(i')}_{k'}}_{c,c} \ge \frac{\theta}{8q^{(i')}_{k'}}, \tag{5}$$

or some $d \in [0, q^{(i'+2)}_{k'})$ with

$$-A^{q^{(i'+2)}_{k'},M^{(i'+2)}_{k'}}_{d+1,d} \ge \frac{\theta}{8q^{(i')}_{k'}}.$$

Proof. In this proof we again suppress the i', k', u in our notation, and write instead q, M, for $q^{(i')}_{k'}$ and $M^{(i')}_{k'}$, respectively (the terms corresponding to $i'+2$ will come up in the proof later). Now, for every $c, r \in [0,q]$,

$$\sum_{c=0}^{q-1} A^{q,M}_{c+r,c} = \frac{1}{qM}\sum_{m=1}^{qM} \gamma(m)\cdot(a(m+r)+\ldots+a(m+r+q-1)) + O\left(\frac{1}{M}\right)$$

So,

$$qM \cdot \sum_{c=0}^{q-1} A^{q,M}_{c,c} = \sum_{m=1}^{qM} \gamma(m)\cdot(a(m)+\ldots+a(m+q-1)) + O(q)$$

and

$$(q-1)\left[\frac{qM}{q-1}\right]\sum_{c=1}^{q-1} A^{q-1,\left[\frac{qM}{q-1}\right]}_{c+1,c} = \sum_{m=1}^{qM}\gamma(m)\cdot(a(m+1)+\ldots+a(m+q-1)) + O\left(q^2\right)$$

Combining the last two displayed equations,

$$qM\cdot\sum_{c=0}^{q-1} A^{q,M}_{c,c} - (q-1)\left[\frac{qM}{q-1}\right]\sum_{c=1}^{q-1} A^{q-1,\left[\frac{qM}{q-1}\right]}_{c+1,c} = \sum_{m=1}^{qM}\gamma(m)a(m) + O\left(q^2\right) \ge \theta/2\cdot qM + O\left(q^2\right).$$

It follows that, assuming q is large enough, and via (4)

$$\sum_{c=0}^{q-1} A^{q,M}_{c,c} - \sum_{d=1}^{q-1} A^{q-1,\left[\frac{qM}{q-1}\right]}_{d+1,d} \ge \theta/2 - O\left(\frac{q}{M}\right) \ge \theta/2 - O\left(\frac{1}{q^2}\right) \ge \theta/4.$$

Recalling our definition of $q^{(i'+2)}_{k'}$ and $M^{(i'+2)}_{k'}$, this implies the Lemma. □

2.4 Construction of the point and system as in Theorem 1.2

Recalling Lemma 2.8, by perhaps moving to a further subseqeunce, we may assume that the inequality from Lemma 2.8 is always given by the term corresponding to $q_{k'}^{(i'+2u)}$ where $u = u(j)$ is either 0 or 1, and both $i' = i(j)$ and u are assumed to be constant in j. Let us denote this constant value $i' + 2u \in \{0, 1, 2, 3\}$ by ℓ. Recalling Definition 2.5, and passing to a subsequence if needed, we assume that the map $j \mapsto k(j) = k'$ is injective.

We now construct a point $x^{(\ell)} \in P^{(\ell)} \times \{-1, 0, 1\}^{\mathbb{Z}} \subseteq X_\ell$ as follows: For every $j \in \mathbb{N}$ and $q_{k(j)}^{(\ell)} \leq n < q_{k(j)+1}^{(\ell)}$, $x^{(\ell)}(n) = x(n)$ where x is the element as in Lemma 2.6 (if $u = 0$) or Remark 2.7 (if $u = 1$), corresponding to j, ℓ as in the paragraph above, and either $r = c$ and c (if $u = 0$) or $r = d + 1$ and $c = d$ (if $u = 1$) yielding the inequality from Lemma 2.8. Note that here we need the map $j \mapsto k(j)$ to be injective so this is well defined (i.e. the intervals $[q_{k(j)}^{(\ell)}, q_{k(j)+1}^{(\ell)})$ don't overlap). Note that so far we have only specified the digits $n \in \bigcup_{j \in \mathbb{N}} [q_{k(j)}^{(\ell)}, q_{k(j)+1}^{(\ell)})$, and (since we have passed to a subsequence) it is possible that this union does not cover all of $\mathbb{N}$. So, for all digits not covered we make some choice that ensures $x^{(\ell)} \in P^{(\ell)} \times \{-1, 0, 1\}^{\mathbb{Z}}$. Note that by Lemma 2.6 and the construction of $P^{(\ell)}$, such a choice is readily available.

We now take our space to be

$$X := X_0 \times X_1 \times X_2 \times X_3 \times \{0, 1, 2, 3\}, \tag{6}$$

with the self-mapping $\hat{T} \in \mathcal{C}(X)$ being

$$\hat{T}(p^{(0)}, p^{(1)}, p^{(2)}, p^{(3)}, i) = (Tp^{(0)}, Tp^{(1)}, Tp^{(2)}, Tp^{(3)}, i).$$

The function $f \in \mathcal{C}(X)$ is taken to be

$$f((y^{(0)}, z^{(0)}), (y^{(1)}, z^{(1)}), (y^{(2)}, z^{(2)}), (y^{(3)}, z^{(3)}), i) = z_0^{(i)}.$$

We next choose our point x to be any $x \in X$ such that: Its projection to X_ℓ is $x^{(\ell)}$, and its projection to $\{0, 1, 2, 3\}$ is ℓ.

We now prove part (1) of Theorem 1.2 via the following two claims:

Claim 2.9. *We have $h(X, \hat{T}) = 0$.*

Proof. By Claim 2.4 each factor in the product space X has zero entropy, which implies the assertion via standard arguments. □

Claim 2.10. *For all j large enough,*

$$\frac{1}{N_j} \sum_{n=1}^{N_j} f(\hat{T}^n x) a(n) \geq \theta \cdot \tau(N_j).$$

In particular,

$$\limsup_{N \to \infty} \frac{\frac{1}{N} \sum_{n=1}^{N} f(\hat{T}^n x) a(n)}{\tau(N)} > 0.$$

Proof. Fix j large, and let us write N, q, M, x, suppressing the dependence on k', ℓ, j (except in parts of the proof where we wish to emphasize this dependence). Note that

$$qM \in [N - q, N].$$

Now:

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} f(\hat{T}^n x)a(n) &= \frac{1}{qM}\sum_{n=1}^{qM} f(\hat{T}^n x)a(n) + O(\frac{1}{M}) \\
&= \frac{1}{qM}\left(\sum_{n=1}^{q-1} f(\hat{T}^n x)a(n) + \sum_{n=q}^{qM} f(\hat{T}^n x)a(n)\right) + O(\frac{1}{M}) \\
&= \frac{1}{qM}\left(\sum_{n=1}^{q-1} f(\hat{T}^n x)a(n) + \sum_{n=1}^{qM} g(T^n x^{(\ell)})a(n) - \sum_{n=1}^{q-1} g(T^n x^{(\ell)})a(n)\right) \\
&\quad +O(\frac{1}{M}) \\
&= \frac{1}{q_{k'}^{(\ell)} M_{k'}^{(\ell)}} \sum_{n=1}^{q_{k'}^{(\ell)} M_{k'}^{(\ell)}} g(T^n x^{(\ell)})a(n) + O(\frac{1}{M_{k'}^{(\ell)}}) \\
&\geq \frac{\theta}{8q_{k'}^{(i')}} + O\left(\frac{q_{k'}^{(\ell)}}{M_{k'}^{(\ell)}}\right)
\end{aligned}$$

Note that in the third equality we are again using Lemma 2.3 in a similar fashion to the proof of Lemma 2.6, which is allowed since $x^{(\ell)} \in P^{(\ell)} \times \{-1,0,1\}^{\mathbb{Z}}$. For the last inequality we are using Lemmas 2.8 and 2.6 along with the definition of x.

We conclude that

$$\frac{1}{N_j}\sum_{n=1}^{N_j} f(\hat{T}^n x)a(n) \geq \frac{\theta}{8q_{k'}^{(i')}} + O\left(\frac{q_{k'}^{(\ell)}}{M_{k'}^{(\ell)}}\right).$$

By (4),

$$O\left(\frac{q_{k'}^{(\ell)}}{M_{k'}^{(\ell)}}\right) \leq O\left(\left(\frac{1}{q_{k'}^{(i')}}\right)^2\right),$$

and so, as long as j is large enough,

$$\frac{1}{N_j}\sum_{n=0}^{N_j-1} f(\hat{T}^n x)a(n) \geq \frac{\theta}{16q_{k'}^{(i')}}.$$

Finally, it follows from our choice of N_j that N_j is larger than the element of the sequence $q_k/3$ that comes after $q_{k'}^{(i')}/3$. So, by the choice of the sequence q_k,

$$\frac{1}{16q_{k'}^{(i')}} \geq \tau(N_j).$$

Combining the last two displayed equations implies the Claim.

□

3 Proof of Theorem 1.2 Part (2)

In this Section we prove Part (2) of Theorem 1.2. That is, we show that the system $(X, \hat{T})$ given in (6) satisfies the Möbius disjointness conjecture (1). The proof will be an application of Matomäki-Radziwiłł's bound [13] on averages of multiplicative functions along short intervals, which has recently become a standard tool to establish Möbius disjointness for systems with strong periodic behaviour.

Denote a point $x \in X$ as

$$(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, i), \text{ where } x^{(\ell)} = (y^{(\ell)}, z^{(\ell)}).$$

For each $p = (y, z) \in \{-1, 0, 1\}^{\mathbb{N}} \times \{-1, 0, 1\}^{\mathbb{Z}}$ and $M \in \mathbb{N}$, denote by $[p]_M$ the truncation

$$[p]_M := \big((y_1, \cdots, y_M), (z_{-M}, \cdots, z_M)\big).$$

Write $\mathcal{C}_M(X)$ for the space of cylinder functions $f(x)$ that only depends on $([x^{(\ell)}]_M)_{0\le\ell\le3}$ and the fifth coordinate $i \in \{0, 1, 2, 3\}$. Then $\bigcup_{M=1}^{\infty} \mathcal{C}_M(X)$ is dense in $\mathcal{C}(X)$ with respect to C^0 norm. In consequence, it suffices to verify (1) for all cylinder functions $f \in \mathcal{C}_M(X)$ for every M.

The main technical Lemma that we need is the following:

Lemma 3.1. *For all $0 \le \ell \le 3$ and $M, H \in \mathbb{N}$ and $x \in X$, there exists a set $\Lambda^{(\ell)}(M, H, x) \subseteq \mathbb{N}$ that satisfies:*

1. $\lim_{N\to\infty} \frac{1}{N} \#\big(\{1, \cdots, N\} \cap \Lambda^{(\ell)}(M, H, x)\big) = 1.$
2. *For all $n \in \Lambda^{(\ell)}(M, H, x)$, $[T^{n+h}x^{(\ell)}]_M$ is constant for $0 \le h \le H - 1$.*

Proof. Since

$$x^{(\ell)} \in X_\ell = \mathrm{cl}\left(\bigcup_{b\in\mathbb{N}_0} T^b(P^{(\ell)} \times \{-1, 0, 1\}^{\mathbb{Z}})\right),$$

for each ℓ and all $N \in \mathbb{N}_0$, there exists $x^{(N,\ell)} \in \bigcup_{b\in\mathbb{N}_0} T^b(P^{(\ell)} \times \{-1, 0, 1\}^{\mathbb{Z}})$ such that

$$[x^{(N,\ell)}]_N = [x^{(\ell)}]_N \text{ for all } n \le N.$$

We also choose $b^{(N,\ell)} \in \mathbb{N}_0$ and $\tilde{x}^{(N,\ell)} \in P^{(\ell)} \times \{-1, 0, 1\}^{\mathbb{Z}}$ such that $x^{(N,\ell)} = T^{b^{(N,\ell)}}\tilde{x}^{(N,\ell)}$.

Then for $1 \le n \le N$ and $0 \le h \le H - 1$,

$$[T^{n+h}x^{(\ell)}]_M = [T^{n+h}x^{(N+H+M,\ell)}]_M = [T^{n+b^{(N+H+M,\ell)}+h}\tilde{x}^{(N+H+M,\ell)}]_M.$$

Therefore, by part (1) of Lemma 2.1, $[T^{n+h}x^{(\ell)}]_M$ is constant for $0 \le h \le H - 1$ if

$$\Pi_1\tilde{x}^{(N+H+M,\ell)}(n + b^{(N,\ell)} + h') = 0, \text{ for all } 0 \le h' \le H + M - 1. \tag{7}$$

Since $\tilde{x}^{(N+H+M,\ell)} \in P^{(\ell)} \times \{-1, 0, 1\}^{\mathbb{Z}}$, for every $k \in \mathbb{N}$ there is some $0 \le r_k^{(\ell)} \le q_k^{(\ell)} - 1$ such that

$$\Pi_1\tilde{x}^{(N+H+M,\ell)}(n') = s_k^{(\ell)}(n' - r_k^{(\ell)}) \text{ for } q_k^{(\ell)} \le n' < q_{k+1}^{(\ell)}.$$

In particular, $\Pi_1\tilde{x}^{(N+H+M,\ell)}(n') = 0$ for all $q_k^{(\ell)} \le n' < q_{k+1}^{(\ell)}$ with $n' \not\equiv r_k^{(\ell)} (\mathrm{mod}\ q_k^{(\ell)})$.

It follows that for each k, (7) holds on the set

$$\begin{aligned}\Lambda_{N,k}^{(\ell)}(M, H, x) :=\{&1 \le n \le N : q_k^{(\ell)} \le n + b^{(N+H+M,\ell)} \le q_{k+1}^{(\ell)} - H - M;\\ &n + b^{(N+H+M,\ell)} \not\equiv r_k^{(\ell)} - H - M + 1, \cdots, r_k^{(\ell)} - 1, r_k^{(\ell)} (\mathrm{mod}\ q_k^{(\ell)})\}.\end{aligned}$$

Set $\Lambda_N^{(\ell)}(M, H, x) = \bigcup_{k=1}^{\infty} \Lambda_{N,k}^{(\ell)}(M, H, x) \subseteq \{1, \cdots, N\}$. Then $[T^{n+h}x^{(\ell)}]_M$ is constant for $0 \le h \le H - 1$ if $n \in \Lambda^{(\ell)}(M, H, x)$.

Finally,

$$\lim_{N\to\infty} \frac{1}{N}\#\big(\{1,\cdots,N\}\cap \Lambda_N^{(\ell)}(M,H,x)\big) = 1$$

because of the following facts: H and M are fixed, $b^{(N+H+M,\ell)} \geq 0$, $\lim_{k\to\infty} q_k^{(\ell)} = \infty$ and $\lim_{k\to\infty} \frac{q_{k+1}^{(\ell)}}{q_k^{(\ell)}} = \infty$. We conclude the proof by defining

$$\Lambda^{(\ell)}(M,H,x) := \bigcup_{N=1}^{\infty} \Lambda_N^{(\ell)}(M,H,x).$$

□

Corollary 3.2. *For all $M,H \in \mathbb{N}$ and $x \in X$, there exists a set $\Lambda(M,H,x) \subseteq \mathbb{N}$ that satisfies:*

1. $\lim_{N\to\infty} \frac{1}{N}\#\big(\{1,\cdots,N\}\cap \Lambda(M,H,x)\big) = 1$.
2. *For all $f \in \mathcal{C}_M(X)$ and any given $n \in \Lambda(M,H,x)$, $f(\hat{T}^{n+h}x)$ is constant for $0 \leq h \leq H-1$.*

Proof. Let $\Lambda^{(\ell)}(M,H,x)$ be as in Lemma 3.1, and set

$$\Lambda(M,H,x) := \bigcap_{0\leq \ell\leq 3} \Lambda^{(\ell)}(M,H,x) \subset \mathbb{N}.$$

Then clearly we still have

$$\lim_{N\to\infty} \frac{1}{N}\#\big(\{1,\cdots,N\}\cap \Lambda(M,H,x)\big) = 1.$$

Next, let $f \in \mathcal{C}_M(X)$. Since $f(\hat{T}^{n+h}x)$ only depends on $\big([T^{n+h}x^{(\ell)}]_M\big)_{0\leq\ell\leq 3}$ and the i coordinate (that does not change when we apply $\hat{T}$), given $n \in \Lambda^{(\ell)}(i,M,H,x)$, it is constant for $0 \leq h \leq H-1$ by Lemma 3.1. □

We are now ready to establish Möbius disjointness:

Proof of Theorem 1.2 Part (2). As remarked in the beginning of this Section, we may assume $f \in \mathcal{C}_M(X)$ for some M and $|f| \leq 1$. Let $x \in X$. Then for a fixed H, as $N \to \infty$,

$$\begin{aligned}
\Big|\frac{1}{N}\sum_{n=1}^{N} f(\hat{T}^n x)\mu(n)\Big| &= \Big|\frac{1}{N}\sum_{n=1}^{N}\frac{1}{H}\sum_{h=0}^{H-1} f(\hat{T}^{n+h}x)\mu(n)\Big| + O(\frac{H}{N}) \\
&= \Big|\frac{1}{N}\sum_{\substack{1\leq n\leq N\\ n\in\Lambda(M,H,x)}}\frac{1}{H}\sum_{h=0}^{H-1} f(\hat{T}^{n+h}x)\mu(n)\Big| + o_H(1) + O(\frac{H}{N}) \\
&\leq \frac{1}{N}\sum_{\substack{1\leq n\leq N\\ n\in\Lambda(M,H,x)}}\Big|\frac{1}{H}\sum_{h=0}^{H-1} f(\hat{T}^{n+h}x)\mu(n+h)\Big| + o_H(1) + O(\frac{H}{N})
\end{aligned}$$

Here $o_H(1)$ stands for a quantity that tends to 0 as $N \to \infty$ for a fixed H.

By Corollary 3.2, $f(\hat{T}^{n+h}x) = f(\hat{T}^n x)$ for every $n \in \Lambda(M,H,x)$ and $0 \le h \le H-1$. So,

$$\begin{aligned}
\Big|\frac{1}{N}\sum_{n=1}^{N} f(\hat{T}^n x)\mu(n)\Big| \le & \frac{1}{N}\sum_{\substack{1\le n\le N\\ n\in\Lambda(M,H,x)}} \Big|\frac{1}{H}\sum_{h=0}^{H-1} f(\hat{T}^n x)\mu(n+h)\Big| + o_H(1) + O(\frac{H}{N})\\
\le & \frac{1}{N}\sum_{\substack{1\le n\le N\\ n\in\Lambda(M,H,x)}} \Big|\frac{1}{H}\sum_{h=0}^{H-1} \mu(n+h)\Big| + o_H(1) + O(\frac{H}{N})\\
\le & \frac{1}{N}\sum_{n=1}^{N} \Big|\frac{1}{H}\sum_{h=0}^{H-1} \mu(n+h)\Big| + o_H(1) + O(\frac{H}{N})\\
= & O\Big((\frac{1}{\log H})^{0.01} + (\frac{\log H}{\log N})^{0.01}\Big) + o_H(1) + O(\frac{H}{N}).
\end{aligned}$$

The last step is given by [13, Theorem 1].

By letting $H \to \infty$ first, and then $N \to \infty$ for each fixed H, we see that

$$\frac{1}{N}\sum_{n=1}^{N} f(\hat{T}^n x)\mu(n) = o(1) \text{ as } N \to \infty.$$

□

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