THE NORMAL SUBGROUP THEOREM THROUGH MEASURE RIGIDITY

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Abstract. We present an expository proof of Margulis's Normal Subgroup Theorem and Measurable Factor Theorem using tools of measure rigidity for actions of higher-rank abelian groups in homogeneous dynamics.

1. Introduction and main results

1.1. **Introduction.** We present an expository proof of Margulis's *Normal Subgroup Theorem*, Theorem 1.1 below, which appeared in [15, 17] as translated in [16, 18]. For certain discrete subgroups Γ (namely, for irreducible lattices in higher-rank semisimple Lie groups and for some more general groups), the Normal Subgroup Theorem asserts that any normal subgroup N of Γ is either of finite index in Γ or is contained in the center of Γ.

The proof of the Normal Subgroup Theorem follows in two steps: First one establishes that Γ/N has Kazhdan's property (T). When all simple factors of G have higher (real) rank this fact is well known. When Γ is irreducible and G has rank-1 factors, additional arguments are needed to show non-central normal subgroups of Γ have property (T); these appear as [17, Theorem 1.3.2, Theorem 1.4] combined with Margulis's Arithmeticity Theorem. The second step in the proof is to show that Γ/N is amenable whenever N is non-central. This follows from Margulis's *Measurable Factor Theorem*, Theorem 1.2 below, which appears as [15, Theorem 1.14.2]. See also [19, Chapter IV] for more general statements and complete proofs of the Normal Subgroup and Measurable Factor Theorems.

Our proof of the Normal Subgroup Theorem follows Margulis's proof. We present an alternative proof of the Measurable Factor Theorem. Margulis's proof of the Measurable Factor Theorem in [15, 19] may be viewed as a result on the rigidity of certain σ-algebras. The proof we give is based on the rigidity of invariant measures for actions of higher-rank abelian groups in homogeneous dynamics. This approach is inspired by arguments from the authors' paper [1]. However, none of the arguments presented here are original to the authors. In particular, the proof we present below is highly derivative of [5–8, 11, 12].

1.2. **Definitions.** We begin with some definitions. Let $\mathfrak{g}$ be a real Lie algebra. Recall that $\mathfrak{g}$ is *semisimple* if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. The (real) *rank* of $\mathfrak{g}$ is, roughly, the

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dimension of the maximal ad-semisimple, abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}$. (See Section 8.1 below for a more precise definition.) A Lie group G is *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple and the *rank* of G is the rank of $\mathfrak{g}$. We will always assume G is connected. A semisimple Lie group admits a bi-invariant, locally finite volume form called the *Haar measure.* A *lattice* in G is a discrete subgroup such that the quotient G/Γ has finite volume.

A semisimple Lie group G has an almost direct product structure into normal subgroups $G = \Pi_{i=1}^{k} G_i$ of positive dimension. When no normal factor G_i is compact, we say a lattice $\Gamma \subset G$ is *irreducible* if, for every proper subset $C \subset \{1, \dots, k\}$, the image of Γ under the natural projection $G \to \Pi_{i \in C} G_i$ is dense; this implies for any normal subgroup $H \subset G$ of positive dimension that $H\Gamma$ is dense in G.

Let G be semisimple and let $G = KAN$ be a choice of Iwasawa decomposition of G. (See Section 8.1 below for details.) In particular, if G has finite center then K is a maximal compact subgroup, A is a maximal connected abelian subgroup whose image under the adjoint representation is $\mathbb{R}$-diagonalizable, and N is a connected subgroup normalized by A whose image under the adjoint representation is unipotent; moreover, A normalizes N. The subgroup A has Lie algebra $\mathfrak{a}$ and dimension the rank of G. Let $M = C_K(A)$ be the centralizer of A in K and let $P = MAN$. Then P is a *minimal parabolic subgroup.* A *parabolic subgroup* of G is a closed subgroup Q containing a minimal parabolic subgroup for some choice of Iwasawa decomposition.

Given a closed subgroup $Q \subset G$, let λ_Q denote the (left) Haar measure on Q. Given a parabolic subgroup $Q \subset G$, we also denote by $\lambda_{Q\backslash G}$ the (right) K-invariant volume form on $Q\backslash G$; this measure is always in the Lebesgue class. Similarly, given a lattice $\Gamma \subset G$, we write $\lambda_{G/\Gamma}$ for the normalized Haar measure on G/Γ.

1.3. Main results.

We denote by $Z(G)$ the center of the group G. In [15,17], Margulis established the following rigidity of normal subgroups of irreducible lattices in higher-rank Lie groups.

Theorem 1.1 (Normal Subgroup Theorem; [19, Theorem IV.4.10, p. 167])**.** *Let G be a connected semisimple Lie group with rank at least 2 and no non-trivial compact factors. Let Γ be an irreducible lattice subgroup. If $N \lhd \Gamma$ is a normal subgroup of Γ then either $N \subset Z(G)$ or N has finite index in Γ.*

In many situations, such as when G is linear, we have that the center $Z(G)$ is finite; in this case, Theorem 1.1 asserts that every normal subgroup of Γ is either finite or co-finite.

The Normal Subgroup Theorem, Theorem 1.1, follows from the following theorem characterizing measurable factors of the right action of Γ on $P\backslash G$. This action does not preserve any Borel probability measure on $P\backslash G$; however, it preserves the Lebesgue measure class $\lambda = \lambda_{P\backslash G}$. Given a standard measure space (X, μ), we say a (left) Borel action of Γ on (X, μ) is *non-singular* if $\gamma_* \mu$ is equivalent to μ for every $\gamma \in \Gamma$. Roughly, Margulis's Measurable Factor Theorem states that if a non-singular left-action of Γ on a measure space (X, μ) is a measurable factor of the right Γ-action on $(P\backslash G, \lambda_{P\backslash G})$, then (X, μ) is measurably isomorphic to $(Q\backslash G, \lambda_{Q\backslash G})$ for some parabolic subgroup $Q \supset P$; moreover, this isomorphism intertwines the left Γ-action on (X, μ) with the right Γ-action on $(Q\backslash G, \lambda_{Q\backslash G})$.

Theorem 1.2 (Measurable Factor Theorem; [19, Corollary IV.2.13, p. 154])**.** *Let G be a connected semisimple Lie group with rank at least 2 and no non-trivial compact factors. Let Γ be an irreducible lattice subgroup.*

Let Γ act on a Borel space X and let $p\colon P\backslash G \to X$ be a Borel map defined $\lambda_{P\backslash G}$-a.e. Assume that p is $\Gamma-$equivariant: for $\lambda_{P\backslash G}$-a.e. g and every $\gamma \in \Gamma$ we have

$$p(Pg\gamma) = \gamma^{-1} \cdot p(Pg)$$

Let $\mu = p_*\lambda_{P\backslash G}$ be the image of $\lambda_{P\backslash G}$ under p. Then there is a parabolic subgroup $Q \supset P$ such that (X, μ) is Γ-equivariantly isomorphic to $(Q\backslash G, \lambda_{Q\backslash G})$: there is a Γ-equivariant isomorphism of measure spaces $H\colon (X, \mu) \to (Q\backslash G, \lambda_{Q\backslash G})$ such that if $\pi\colon P\backslash G \to Q\backslash G$ is the natural map, then the following diagram commutes:

$$\begin{array}{ccc} (P\backslash G, \lambda_{P\backslash G}) & \xrightarrow{p} & (X, \mu) \\ & \searrow^{\pi} & \downarrow H \\ & & (Q\backslash G, \lambda_{Q\backslash G}) \end{array}$$

In the above theorem, the Γ-equivariance of p implies the measure $\mu = p_*\lambda_{P\backslash G}$ is non-singular for the Γ-action on X. Since the action of Γ on $P\backslash G$ is on the right, the Γ-equivariance of the isomorphism $H\colon (X, \mu) \to (Q\backslash G, \lambda_{Q\backslash G})$ asserts that $H(\gamma \cdot x) = H(x) \cdot \gamma^{-1}$ for μ-a.e. x and every $\gamma \in \Gamma$. The isomorphism H in Theorem 1.2 need not be defined everywhere but only on a set of full μ-measure.

A natural setting in which a measurable factor of the Γ-action on $P\backslash G$ appears is stated in Lemma 3.1 below. This forms a key step in the proof of Theorem 1.1 through Lemma 4.1.

One may ask if analogous results hold when the map p in Theorem 1.2 is assumed to be continuous or smooth. In [3], Dani proves a result analogous to Theorem 1.2 for continuous factors, i.e. assuming the map p is a continuous surjection. More recently, Gorodnik and Spatzier studied in [10] smooth factors and (under an additional mild hypothesis) establish a smooth analogue of Theorem 1.2.

2. Representations, property (T), and amenability

To establish Theorem 1.1 we introduce the concepts of amenability and of property (T) groups. We begin with the following definition.

Definition 2.1 (Almost invariant vectors)**.** Let H be a locally compact topological group and let π be a unitary representation of H. We say that π admits *almost-invariant vectors* if, for every $\epsilon > 0$ and every compact subset $C \subset H$, there is a unit vector v with

$$\sup_{h\in C} \| \pi(h)v - v \| < \epsilon.$$

2.1. **Property (T) groups.** We have the following definition.

Definition 2.2 (Property (T) groups)**.** A locally compact topological group H has Kazhdan's *property (T)* if every unitary representation admitting almost invariant vectors has a nontrivial invariant vector.

Remark 2.3 (Facts on property (T) groups)**.** We collect several well-known facts about property (T) groups. See [2, Chapter 1] or [23, Chapter 13] for detailed exposition on property (T).

(1) Compact groups have property (T).
(2) A product group $G_1 \times G_2$ has property (T) if and only if both the factors G_1 and G_2 have property (T).
(3) If G is a connected simple Lie group with real rank at least 2 then G has property (T); more generally, if every almost simple factor of G has real rank at least 2 then G has property (T). (See [19, Corollary III.5.4, p. 130].)
(4) A Lie group G has property (T) if and only if every lattice subgroup Γ of G has property (T). (See [19, Theorem III.2.12, p. 117].)
(5) If a Lie group G has property (T) and if H is a closed normal subgroup of G, then the quotient group G/H has property (T).
(6) Suppose G is a semisimple Lie group with no compact factors and at least one almost-simple factor of G has real rank at least 2. Let Γ be an irreducible lattice. Then for any non-central normal subgroup $N \subset \Gamma$, the quotient Γ/N has property (T). See [19, Theorem III.5.9(B), p. 133]. See also [18, Theorem 1.4].
(7) More generally, suppose a semisimple Lie group G has real-rank at least 2, no compact factors, and that Γ is an irreducible lattice. Then for any non-central normal subgroup $N \subset \Gamma$, the quotient Γ/N has property (T). This follows from [19, Theorem IV.3.9, p. 161], and Margulis' Arithmeticity Theorem, [19, Theorem IV.3.9, p. 161]. See also [18, Theorem 1.3.2].

The group $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\sqrt{2}])$ is an irreducible lattice in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. It is well known that $\mathrm{SL}(2,\mathbb{R})$ fails to have property (T). It follows that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and hence Γ fail to have property (T). However, from Remark 2.3(7) above, the quotient Γ/N has property (T) for any non-central normal subgroup $N \subset \Gamma$.

2.2. **Amenability.** Let λ_H be a left-invariant Haar measure on a locally compact topological group H. Then $L^2(H, \lambda_H)$ is a Hilbert space and the action of H on itself by left translation induces a unitary representation of H on $L^2(H, \lambda_H)$ called the left-regular representation.

Definition 2.4 (Amenability)**.** A locally compact topological group H is *amenable* if the left-regular representation of H in $L^2(H, \lambda_H)$ admits almost invariant vectors.

Examples of amenable groups include all compact groups and all abelian, nilpotent, or solvable Lie groups.

We recall an equivalent notion of amenability that we use in the sequel. See, for instance, [19, (5.5.1), p. 77], [2, Appendix G], or [23, §12.3] for other characterizations of amenability.

Lemma 2.5 (Equivalent characterization of amenability)**.** *H is amenable if and only if any continuous action of H on any compact metric space admits an invariant Borel probability measure.*

2.3. **Property (T) and amenability.** We have the following well-known fact.

Lemma 2.6. *If a locally compact topological group H has property (T) and is amenable then H is compact. In particular, if a countable discrete group has property (T) and is amenable then it is finite.*

Proof. Suppose the left-regular representation of H in $L^2(H, \lambda_H)$ admits almost invariant vectors. If H has property (T) then this representation admits a nontrivial invariant vector. Such a vector coincides with a non-zero constant function $\varphi\colon H \to \mathbb{C}$. However, if a non-zero constant function φ is an element of $L^2(H, \lambda_H)$ then λ_H must be finite whence H is compact. □

3. Suspension space, induced G-action, and Furstenburg's lemma

We present a key construction on which our proof of Theorem 1.2 depends. We also recall a classical result of Furstenburg and give a proof based on this construction.

3.1. **Suspension space and induced G-action.** As above, let G be a semisimple Lie group and let Γ be a lattice subgroup. Consider a continuous action of Γ on a compact metric space X. We recall a standard construction from which we induce a continuous G-action on an auxiliary space.

On $G \times X$ consider the right action of Γ given by

$$(g, x) \cdot \gamma = (g\gamma, \gamma^{-1} \cdot x)$$

and a left action of G given by

$$g' \cdot (g, x) = (g'g, x).$$

These actions commute. Let X^Γ denote the quotient

$$X^\Gamma = (G \times X)/\Gamma$$

by the Γ action. The G-action on $G \times X$ descends to a G-action on X^Γ. Writing $[g, x]$ for the Γ-equivalence class of $(g, x) \in G \times X$ in X^Γ, we have $g' \cdot [g, x] = [g'g, x]$.

Note that X^Γ has a fiber-bundle structure over G/Γ with fibers homeomorphic to X:

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^\Gamma \\ & & \downarrow \pi \\ & & G/\Gamma \end{array}$$

The G-action on X^Γ fibers over the G-action on G/Γ. If Γ is cocompact in G then X^Γ is compact. If Γ is non-uniform, then we may equip X^Γ with a metric such that X^Γ is a complete, second countable, locally compact metric space and such that the projection $X^\Gamma \to G/\Gamma$ is distance non-increasing with respect to some fixed choice of right-invariant distance on G.

3.2. **Furstenburg's Lemma.** We have the following lemma due to Furstenburg; see [9, Theorem 15.1]. Let Γ act continuously on a compact metric space X. Let $\mathcal{P}(X)$ denote the space of Borel probability measures on X equipped with the weak-$*$ topology. The Γ-action on X naturally induces a continuous action of Γ on $\mathcal{P}(X)$: given $\mu \in \mathcal{P}(X)$, $\gamma_* \mu \in \mathcal{P}(X)$ is the Borel measure

$$\gamma_* \mu(B) = \mu(\gamma^{-1} \cdot B).$$

Lemma 3.1 (Furstenburg's lemma)**.** *There exists a Borel measurable function*

$$h \colon G \to \mathcal{P}(X)$$

such that

(1) $h(g\gamma^{-1}) = \gamma_* h(g)$ for λ_G-a.e. g and every $\gamma \in \Gamma$;
(2) $h(pg) = h(g)$ for every $p \in P$ and λ_G-a.e g.

In particular, h descends to a Γ-equivariant map $h \colon P\backslash G \to \mathcal{P}(X)$ defined $\lambda_{P\backslash G}$-a.e.

Proof. Let X^Γ be the suspension space associated to the Γ-action on X. Let $P = MAN$ be a minimal parabolic subgroup of G. As P is a compact extension of a solvable group, P is amenable. Although X^Γ need not be compact, the set of Borel probability measures on X^Γ projecting to the normalized Haar measure $\lambda_{G/\Gamma}$ on G/Γ is a compact, P-invariant subset of the space of Borel probability measures on X^Γ. Since P is amenable, by a slight extension of the characterization of amenability in Lemma 2.5, there exists a P-invariant Borel probability measure ν on X^Γ projecting to the normalized Haar measure $\lambda_{G/\Gamma}$ on G/Γ.

Fix such a P-invariant Borel probability measure ν on X^Γ. There exists a unique lift $\tilde{\nu}$ of ν to a locally finite Borel measure on $G \times X$. The measure $\tilde{\nu}$ is a Γ-invariant, P-invariant, Borel measure that is finite on compact sets. The partition of $G \times X$ into elements of the form $\{g\} \times X$ is measurable and hence admits a family of conditional measures $\tilde{\nu}_g$ (see Definition 6.1 and Lemma 6.2 below) parameterized by $g \in G$. Identifying each $\{g\} \times X$ with X, we view each $\tilde{\nu}_g$ as a Borel probability measure on X and obtain a measurable map $h \colon G \to \mathcal{P}(X)$ given by $h \colon g \mapsto \tilde{\nu}_g$.

By the P-invariance of $\tilde{\nu}$, we have $\tilde{\nu}_g = \tilde{\nu}_{pg}$ for $p \in P$ and hence h descends to a well-defined function $h \colon P\backslash G \to \mathcal{P}(X)$ and (2) follows. Moreover, since the lifted measure $\tilde{\nu}$ on $G \times X$ is Γ-invariant, we obtain Γ-equivariance of the measures $\{\tilde{\nu}_g\}$,

$$\nu_{g\gamma} = \gamma_*^{-1}\tilde{\nu}_g,$$

and (1) follows. □

4. The Measurable Factor Theorem implies the Normal Subgroup Theorem

The proof of Theorem 1.1 follows immediately from the following lemma which we derive from Theorem 1.2 later in this section.

Lemma 4.1 (c.f. [15, Theorem 2.7])**.** *Let G be a connected semisimple Lie group with real rank at least 2 and no non-trivial compact factors. Let Γ be an irreducible lattice subgroup and let N be normal subgroup of Γ. Then either $N \subset Z(G)$ or Γ/N is amenable.*

Proof of the Normal Subgroup Theorem, Theorem 1.1. Let N be a non-central normal subgroup of Γ. As discussed in Remark 2.3, the quotient $H = \Gamma/N$ has property (T). By Lemma 4.1, if N is non-central then $H = \Gamma/N$ is amenable and hence finite by Lemma 2.6. Thus N is of finite index in Γ whenever N is non-central. □

To establish Lemma 4.1, we use the characterization of amenability in Lemma 2.5. Fix a compact metric space X. Consider an action of $H = \Gamma/N$ by homeomorphisms of X. Note that the action of H induces an action of Γ for which every element in N acts as the identity transformation. Assuming that N is non-central, we will show there exists an invariant Borel probability measure for this action; as X was arbitrary, it follows from Lemma 2.5 that H is amenable.

Proof of Lemma 4.1. Let $Y = \mathcal{P}(X)$ denote the set of Borel probability measures on X equipped with the weak-$*$ topology. The continuous action of Γ on X induces a continuous action of Γ on Y. By Lemma 3.1, we obtain a Γ-equivariant Borel measurable map $h\colon P\backslash G \to Y$ defined $\lambda_{P\backslash G}$-a.e. Let $\mu = h_*\lambda_{P\backslash G}$. Then μ is a non-singular measure for the action of Γ on Y and (Y, μ) is a Γ-equivariant factor of the Γ-action on $(P\backslash G, \lambda_{P\backslash G})$. By Theorem 1.2, there is a parabolic subgroup Q and a Γ-equivariant, measurable isomorphism $H\colon (Q\backslash G, \lambda_{Q\backslash G}) \to (Y, \mu)$ such that $h = H \circ \pi$ where $\pi\colon P\backslash G \to Q\backslash G$ is the natural projection.

We claim that $Q = G$ whenever N is non-central. In this case, the quotient $Q\backslash G$ is a singleton whence the measure μ on $Y = \mathcal{P}(X)$ is a point-mass, $\mu = \delta_{\mu_0}$, for some $\mu_0 \in Y = \mathcal{P}(X)$. It follows that μ_0 is a fixed point for the Γ-action on $\mathcal{P}(X)$ whence μ_0 is a Γ-invariant Borel probability measure on X.

To complete the proof, it suffices to show that $Q \neq G$ implies $N \subset Z(G)$. Recall that $N \subset \Gamma$ acts trivially on X and hence also acts trivially on $Y = \mathcal{P}(X)$. By the measurable identification of (Y, μ) with $(Q\backslash G, \lambda_{Q\backslash G})$, we have that N acts trivially as a group of measurable transformations of $(Q\backslash G, \lambda_{Q\backslash G})$; as N acts continuously on $Q\backslash G$ and as $\lambda_{Q\backslash G}$ has full support, we have that N acts on $Q\backslash G$ by the identity homeomorphism.

Let $L \subset G$ denote the kernel of the right action of G on $Q\backslash G$; that is,

$$L = \{h \in G : Qgh = Qg \text{ for all } g \in G\}.$$

We have that L is a closed normal subgroup of G. If $L \neq G$ then G may be written as an almost direct product $G = H \cdot L$ for some normal subgroup $H \subset G$ of positive dimension. Since $N \subset L$ and H commutes with L, we have that H is contained in $N_G(N)$, the normalizer of N in G. Moreover, as N is normal in Γ we have $\Gamma \subset N_G(N)$. We thus have

$$\overline{H \cdot \Gamma} \subset N_G(N).$$

However, since Γ is irreducible, we have $\overline{H \cdot \Gamma} = G$. It follows that N is a discrete normal subgroup of G. This implies $N \subset Z(G)$ by a standard fact we recall in Lemma 4.2 below. □

To complete the proof of Lemma 4.1, we recall the following well-known fact and its proof.

Lemma 4.2. *If N is a discrete normal subgroup of G then N is central.*

Proof. Fix $n \in N$. Since N is discrete and normal, there is a compact neighborhood of the identity, $C \subset G$, such that $gng^{-1} = n$ for all $g \in C$. Since G is connected, C generates G and it follows that $gng^{-1} = n$ for all $g \in G$. □

5. Preliminaries and reformulation of Theorem 1.2

Let X be a standard Borel space. Let Γ be as in Theorem 1.2 and consider a Borel action of Γ on X. In the setting of the proof of Theorem 1.1, the natural action used in the proof of Lemma 4.1 was a continuous action. We have the following which, in the abstract setting of Theorem 1.2, allows us to assume that the action is continuous.

Lemma 5.1 (See [22, Theorem 3.2.])**.** *There exists a compact metric space Z, a continuous Γ-action on Z, and an injective, Γ-equivariant Borel map $\iota\colon X \to Z$.*

Pushing forward the measure on X to a measure on Z, we obtain an almost surjective, Γ-equivariant function $p\colon P\backslash G \to Z$. Replacing X with Z, we may thus assume for the remainder that the Γ-action on X is continuous.

We follow the notation of Theorem 1.2. Lift the Γ-equivariant measurable map $p\colon P\backslash G \to X$ to a Γ-equivariant map $\hat{p}\colon G \to X$,

$$\hat{p}(g) = p(Pg).$$

Let

$$Q := \{g \in G : \hat{p}(gx) = \hat{p}(x) \text{ for } \lambda_G\text{-a.e. } x \in G\}.$$

We have $P \subset Q$. Moreover, the definition implies that Q is a subgroup of G. Indeed if $g_1, g_2 \in Q$ then there are full measure subsets $R_1, R_2 \subset G$ such that $\hat{p}(g_i x) = \hat{p}(x)$

for all $x \in R_i$; then $R = (g_2^{-1} \cdot R_1) \cap R_2$ has full measure in G and for $x \in R$ we have

$$\hat{p}(g_1 g_2 x) = \hat{p}(g_2 x) = \hat{p}(x)$$

whence $g_1 g_2 \in Q$. Similarly, the set $R' = g_1 \cdot R_1$ has full measure in G and for $x \in R'$,

$$\hat{p}(x) = \hat{p}(g_1 g_1^{-1} x) = \hat{p}(g_1^{-1} x)$$

whence $g_1^{-1} \in Q$.

Although not clear from the above definition, it will follow from observations below that Q is a closed subgroup. In particular, Q is a parabolic subgroup of G.

By definition of Q, the function $p\colon P\backslash G \to X$ in Theorem 1.2 descends to a well-defined, Γ-equivariant function $Q\backslash G \to X$. To establish Theorem 1.2, it remains to show that induced function $Q\backslash G \to X$ is $\lambda_{Q\backslash G}$-a.s. injective. That is, we show for a full-measure subset of G that the preimages of $\hat{p}\colon G \to X$ are Q-orbits. In particular, the proof of Theorem 1.2 follows from the following.

Lemma 5.2. *There exists a full λ_G-measure subset $\hat{R} \subset G$ such that if $g \in \hat{R}$ then*

$$\hat{p}^{-1}(\hat{p}(g)) \cap \hat{R} \subset Qg.$$

To begin the proof of Lemma 5.2, we construct a Borel probability measure ν on X^Γ. To construct this measure ν, let $\hat{P}\colon G \to G \times X$ denote the inclusion of G into the graph of $\hat{p}$; that is,

$$\hat{P}(g) = (g, \hat{p}(g)).$$

Let $\tilde{\nu} = \hat{P}_* \lambda_G$ denote the image of the Haar measure on G under $\hat{P}$. Then $\tilde{\nu}$ is a locally finite Borel measure on $G \times X$. By the Γ-equivariance of $\hat{p}$, the measure $\tilde{\nu}$ is right Γ-invariant and hence descends to a finite Borel measure ν on X^Γ which we normalize to be a probability measure.

By the definition of Q we have that $Q \subset G$ coincides with the stabilizers of $\tilde{\nu}$ and ν. In particular, this shows that Q is a closed subgroup of G.

Lemma 5.2 may be reformulated in terms of the *leaf-wise measures* of ν along orbits of certain subgroups of G acting on X^Γ. See Section 6.3 below for details on leaf-wise measures. Here we simply describe the properties that we use. Let N^- denote the subgroup opposite to N; that is, if N is generated by positive root spaces then N^- is generated by negative root spaces (see Section 8.1 below). Then PN^- is a dense open subset of G.

Associated to each subgroup $H \subset G$ (for which ν-almost every H-orbit on X^Γ is free) we construct in Section 6.3 below a measurable family of locally finite (hence Radon) Borel measures $\{\nu_x^H : x \in X\}$ on H called the *leaf-wise* measures of ν associated to the subgroup H.

Given $x \in X^\Gamma$ with free H-orbit, let

$$\Phi_x\colon H \to X, \quad \Phi_x\colon h \mapsto h \cdot x$$

be the canonical parametrization of the orbit $H \cdot x$. We may then push forward each measure ν_x^H to a Borel (in the intrinsic orbit topology on $H \cdot x$) measure $(\Phi_x)_* \nu_x^H$

on the orbit $H \cdot x$. Recall that two locally finite Borel measures μ_1, μ_2 on H are *proportional*, written $\mu_1 \propto \mu_2$, if there is $c > 0$ such that

$$\mu_1(B) = c\mu_2(B)$$

for all Borel sets B. The family of leaf-wise measures $\{\nu_x^H : x \in X^\Gamma\}$ on H have the following properties that we use:

(1) If $E \subset X^\Gamma$ is a Borel set then $\nu(E) = 0$ if and only if for ν-a.e. x,
$$(\Phi_x)_* \nu_x^H(E) = 0.$$
(2) There is a full measure subset $E \subset X^\Gamma$ such that for $x \in E$ and $h \in H$ such that $h \cdot x \in E$,
$$(\Phi_{hx}^{-1})_*(\Phi_x)_* \nu_x^H \propto \nu_{h\cdot x}^H.$$

Note that if $y = h \cdot x \in H \cdot x$ then $\Phi_y^{-1} \circ \Phi_x \colon H \to H$ corresponds to right translation by h^{-1}; indeed
$$\Phi_y^{-1} \circ \Phi_x(h') = \Phi_y^{-1}(h'h^{-1}hx) = h'h^{-1}.$$
In particular, property (2) above implies
$$(r_{h^{-1}})_* \nu_x^H \propto \nu_{h\cdot x}^H$$
where $r_h \colon H \to H$ denotes right-translation on H by h.

Write $Q^- := N^- \cap Q$. Lemma 5.2 is equivalent to the following proposition whose proof occupies Sections 6–9.

Proposition 5.3. *There exists a set $R' \subset X^\Gamma$ of full ν-measure such that for $x \in R'$, the measure $\nu_x^{N^-}$ is supported on Q^-. In particular, for $x \in R'$,*
$$\nu_x^{N^-} = \nu_x^{Q^-}.$$

We note that since ν is assumed to be Q^--invariant, we have that $\nu_x^{Q^-}$ is the Haar measure λ_{Q^-} on Q^- for ν-almost every $x \in R'$. (See Claim 6.5 below.)

We show Proposition 5.3 implies Lemma 5.2.

Proof of Lemma 5.2. From Proposition 5.3, the properties of leaf-wise measures discussed above, and Q-invariance of ν we may find a subset $R_0 \subset X^\Gamma$ with $\nu(R_0) = 1$ on which the following properties hold:

(1) For $x \in R_0$, we have $\nu_x^{N^-} = \lambda_{Q^-}$.
(2) For $x \in R_0$ and $h \in N^-$ such that $h \cdot x \in R_0$, we have
$$(\Phi_{hx}^{-1})_*(\Phi_x)_* \nu_x^{N^-} \propto \nu_{h\cdot x}^{N^-};$$
In particular, for all such x and h, we have $h^{-1} \in Q^-$ and hence $h \in Q^-$.
(3) For $x \in R_0$ and λ_Q-almost every $q \in Q$, we have $qx \in R_0$.

Viewing $G \times X$ as a covering space of X^Γ, we lift $R_0 \subset X^\Gamma$ to a Γ-invariant conull subset $\widetilde{R}_0 \subset G \times X$. Let $\hat{R}$ denote the image of $\widetilde{R}_0$ under the projection $G \times X \to G$. We claim Lemma 5.2 holds with this $\hat{R}$.

Take $g \in \hat{R}$ and write $y = \hat{p}(g) \in X$. Then $(g, y) \in \widetilde{R}_0$ and (following the notation from Section 3.1) we have $x = [g, y] \in R_0$. Consider $g' \in \hat{R}$ such that $\hat{p}(g') = y$. Write $h = g'g^{-1}$. Then $h \cdot x = [g', y] \in R_0$.

To complete the proof, we claim $h \in Q$. Every element $h \in G$ can be written in the form $h = q_2^{-1} n q_1$ where $q_i \in Q$ and $n \in N^-$; moreover, for each fixed $h \in G$ there is an open set of $q_1 \in Q$ for which such q_2 and n exist and depend rationally on q_1. In particular, there are $q_1, q_2 \in Q$ such that $q_2 h$ is contained in the N^--orbit of q_1; since $x \in R_0$ and $h \cdot x \in R_0$ and since rational maps are locally Lipschitz (and hence preserve λ_Q-null sets), we may moreover assume we q_1 and q_2 are chosen so that $q_1 \cdot x \in R_0$ and $q_2 h \cdot x \in R_0$.

Since $q_2 h \cdot x \in R_0$ and is contained in the N^--orbit of $q_1 \cdot x$, we have that $q_2 h q_1^{-1}$ is in the support of $\nu^{N^-}_{q_1 \cdot x}$. Since $q_1 \cdot x \in R_0$, it follows that $q_2 h q_1^{-1} \in Q^-$ whence $h \in Q$. ☐

The proof of Proposition 5.3 is carried out in the next 4 sections using tools from smooth ergodic theory and measure rigidity for homogeneous dynamics. The main tools we use are derivative of [5, 6, 11].

6. Conditional and leaf-wise measures

6.1. **Measure theory.** Let $(X, \mathcal{A}, \mu)$ be a complete probability space. We say $(X, \mathcal{A}, \mu)$ is *standard* if X may be equipped with a topology of a Polish space such that $\mathcal{A}$ is the μ-completion of the σ-algebra of Borel set $\mathcal{B}$ in this topology. If $(X, \mathcal{A}, \mu)$ is standard, it is measurably isomorphic to the union of an interval $[0, a)$, $0 \leqslant a \leqslant 1$ equipped with the Lebesgue measure and countably many point-masses.

Let $(X, \mathcal{A}, \mu)$ be a standard probability space. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of X by μ-measurable sets. We say that $\mathcal{P}$ is finer than $\mathcal{Q}$ (or that $\mathcal{Q}$ is coarser than $\mathcal{P}$), written $\mathcal{Q} \prec \mathcal{P}$, if there is a full measure subset $Y \subset X$ such that

$$\mathcal{P}(x) \cap Y \subset \mathcal{Q}(x) \cap Y$$

for all $x \in Y$. Given a partition $\mathcal{P}$, we say a measurable subset $A \in \mathcal{A}$ is $\mathcal{P}$-saturated if for all $x \in A$, $\mathcal{P}(x) \subset A$.

Definition 6.1 (Measurable partitions and the measurable hull)**.** A partition $\mathcal{P}$ of $(X, \mathcal{A}, \mu)$ of a standard probability space is *measurable* if there is a countable collection $\{A_i\}$ of $\mathcal{P}$-saturated sets such that for every $x \in X$ and every $y \notin \xi(x)$ there is A_j such that either $\xi(x) \subset A_j$ and $\xi(y) \subset X \smallsetminus A_j$ or $\xi(x) \subset X \smallsetminus A_j$ and $\xi(y) \subset A_j$.

Given an arbitrary partition $\mathcal{P}$, the *measurable hull* of a partition $\mathcal{P}$ is the finest measurable partition $\mathcal{Q}$ with $\mathcal{Q} \prec \mathcal{P}$.

6.2. **Conditional measures.** We now fix X to be a second countable, locally compact, complete metric space. Let $\mathcal{M}$ denote the set of all Borel probability measures on X. Fix $\mu \in \mathcal{M}$. Then (X, μ) is a standard probability space when equipped with the μ-completion of the Borel σ-algebra.

We have the following standard construction.

Lemma 6.2 (Conditional measures; see [21]). *Given a measurable partition $\mathcal{P}$ of (X, μ), there is a measurable function $X \mapsto \mathcal{M}$, written $x \mapsto \mu_x^{\mathcal{P}}$, with the following properties:*

(1) $\mu_x^{\mathcal{P}}$ is a Borel probability measure on X with $\mu_x^{\mathcal{P}}(\mathcal{P}(x)) = 1$;
(2) for a.e. x and every $y \in \mathcal{P}(x)$, we have $\mu_x^{\mathcal{P}} = \mu_y^{\mathcal{P}}$;
(3) for every bounded Borel function $\varphi \colon X \to \mathbb{R}$,

$$\int \varphi \, d\mu = \int \int \varphi(z) \, d\mu_x^{\mathcal{P}}(z) \, d\mu(x).$$

Moreover, up to a null set, the family $x \to \mu_x^{\mathcal{P}}$ is uniquely determined by the above properties.

6.3. **Leaf-wise measures.** Consider a connected, locally compact topological group H equipped with a right-invariant metric. Suppose H acts continuously (on the left) on the complete, second countable, locally compact metric space X. We will moreover assume the action is *locally free*: for every $x \in X$ there is an open neighborhood $U \subset H$ of the identity on which $h \mapsto h \cdot x$ is injective.

Let μ be a Borel probability measure on X. There is a natural partition of X into the orbits of H which we denote by $\mathcal{H}$. In general (and in most situations of interest here) the partition $\mathcal{H}$ of (X, μ) is not a measurable partition. We describe a procedure that associates to each orbit $H \cdot x \in \mathcal{H}$ a locally finite (in the intrinsic topology on the orbit $H \cdot x$ inherited from H) Borel measure which has similar properties to conditional measures associated to measurable partitions.

We begin with the following definition.

Definition 6.3 (Partitions subordinate to orbits). A measurable partition $\mathcal{P}$ is *subordinate* to the partition $\mathcal{H}$ into H-orbits if, for μ-a.e. $x \in X$, the following hold:

(1) $\mathcal{P}(x) \subset H \cdot x$;
(2) $\mathcal{P}(x)$ contains an open (in the orbit topology) neighborhood of x in $H \cdot x$;
(3) $\mathcal{P}(x)$ is pre-compact (in the orbit topology) in $H \cdot x$.

For simplicity, in what follows we will moreover assume that for μ-almost every $x \in H$, the orbit $H \cdot x$ is free; that is, for μ-a.e. x we assume the map $h \mapsto h \cdot x$ is injective. This holds for all groups H considered in the setting of the proof of Proposition 5.3. For such x, we have a canonical parametrization of the orbit $H \cdot x$ given by $\Phi_x \colon H \to X$, $\Phi_x(h) = h \cdot x$. Recall (as discussed in Section 5) that if $y = h' \cdot x \in H \cdot x$ then

$$\Phi_y^{-1} \circ \Phi_x \colon H \to H$$

corresponds to right translation by ${h'}^{-1}$.

Let $\mathcal{R}(H)$ denote the space of locally finite Borel (hence Radon) measures on H equipped with the standard topology (dual to compactly supported functions). Given $r > 0$, let $B_H(r) \subset H$ denote the ball of radius r in H centered at the identity with respect to the fixed right-invariant metric on H.

Proposition 6.4. *There exists a μ-measurable function $X \to \mathcal{R}(H)$, denoted*

$$x \mapsto \nu_x^H$$

such that the following properties hold:

(1) *ν_x^H is normalized so that $\nu_x^H(B_H(1)) = 1$ for μ-a.e. x.*
(2) *For any Borel set $E \subset X$, $\mu(E) = 0$ if and only if for μ-almost every $x \in X$,*
$$\nu_x^H(\Phi_x^{-1}(E)) = 0.$$
(3) *There exists a subset $E_0 \subset X$ of full μ measure such that for $x \in E_0$ and $h \in H$ with $h \cdot x \in E_0$*
$$(\Phi_x)_* \nu_x^H \propto (\Phi_{h\cdot x})_* \nu_{h\cdot x}^H.$$
(4) *For any measurable partition $\mathcal{P}$ subordinate to the partition into H-orbits, μ-almost every x, and $A \subset \mathcal{P}(x)$ we have*
$$\mu_x^{\mathcal{P}}(A) = \frac{(\Phi_x)_* \nu_x^H(A)}{(\Phi_x)_* \nu_x^H(\mathcal{P}(x))}.$$

Moreover, the above properties uniquely determine the family $\{\nu_x^H\}$ modulo null sets.

For a detailed proof, we refer to [4, Theorem 6.3]. Below, we outline a construction of the leaf-wise measures $\{\nu_x^H\}$.

Proof outline. Fix any $R > 1$. Fix $x \in X$ for which the H-orbit of x is free. There are open neighborhoods $x \in W \subset U$ of x such that

(a) for $y \in U$, the connected component of $H \cdot y \cap U$ containing y is a topologically embedded $\dim(H)$-dimensional disk D_y;
(b) the disks $y \mapsto D_y$ depend continuously on y for $y \in U$;
(c) the partition of U into disks $\{D_y : y \in U\}$ is measurable;
(d) $B_H(R) \cdot y \subset D_y$ for every $y \in W$; in particular, $B_H(R) \cdot x \subset D_x$.

Given $y \in U$, let μ_y^U be the conditional measure for the normalized restriction of μ to U relative to the measurable partition $\{D_y : y \in U\}$ of U. Given any Borel subset $E \subset B_H(R)$, and y satisfying property (d) define

$$\nu_y^H(E) := \frac{\mu_y^U(E \cdot y)}{\mu_y^U(B_H(1) \cdot y)}.$$

We may check that $\nu_y^H(E)$ is defined (modulo μ) independently of the choice of R or U. By a countable exhaustion of the space by partitions of the form $\{D_y : y \in U\}$ as $R \to \infty$ as above, for μ-a.e. y the quantity $\nu_y^H(E)$ is defined for any compact subset $E \subset H$. We then obtain a family of measures $\{\nu_x^H\}$ with the desired properties. ☐

We may write $\mu_x^H := (\Phi_x)_* \nu_x^H$ for the locally finite Borel (with respect to the orbit topology) measure on the orbit $H \cdot x$. The family of measures $\{\mu_x^H\}$ may be more natural to consider geometrically. However, it is more convenient in what follows to consider the family $\{\nu_x^H\}$ as each ν_x^H is supported on H and hence we can compare ν_x^H and ν_y^H for $x \neq y$. For the family $\{\mu_x^H\}$, it only makes sense to compare μ_x^H and μ_y^H when $y = h \cdot x$ for some $h \in H$ (in which case we have $\mu_x^H \propto \mu_y^H$.)

As there is no canonical normalization of each ν_x^H, we write

$$[\nu_x^H] := \{c\nu_x^H : c > 0\}$$

for the projective class of measures on H that are positively proportional to ν_x^H. We have the following straight-forward claim whose proof is a standard exercise.

Claim 6.5. *A Borel probability measure μ on X is H-invariant if and only if for almost every $x \in X$, the projective class of the leaf-wise measure $[\nu_x^H]$ coincides with the projective class of left-Haar measures on H.*

7. Measure rigidity

Let H be a connected Lie group. Equip the Lie algebra $\mathfrak{h}$ of H with an inner product and equip H with an induced a right-invariant metric. Write $\mathrm{Isom}(H)$ for the group of isometries of H. Write $\mathrm{Isom}^H(H) \subset \mathrm{Isom}(H)$ for the subgroup of isometries given by right-translations. We canonically identify $\mathrm{Isom}^H(H)$ with H. Given a locally finite Borel measure ν on H, recall that $[\nu]$ is the equivalence class of measures positively proportional to ν. Write

$$\mathrm{Isom}^H(H; [\nu]) := \{g \in \mathrm{Isom}^H(H) : [g_*\nu] = [\nu]\} = \{g \in \mathrm{Isom}^H(H) : g_*\nu \propto \nu\}$$

and let

$$\mathrm{Isom}^H(H; \nu) := \{g \in \mathrm{Isom}^H(H) : g_*\nu = \nu\}.$$

We have the following elementary fact.

Claim 7.1. *The groups $\mathrm{Isom}^H(H; [\nu])$ and $\mathrm{Isom}^H(H; \nu)$ are closed subgroups of H.*

Let H act continuously on a complete, second countable, locally compact metric space X. Let μ be a Borel probability measure on X. We assume that μ-a.e. H-orbit is free. Write $\mathcal{H}$ for the partition of (X, μ) into H-orbits.

Write $\mathrm{Aut}(X, \mu)$ for the set of bijective, measurable, μ-preserving maps of X. Since (X, μ) is standard, we have $f^{-1} \in \mathrm{Aut}(X, \mu)$ for every $f \in \mathrm{Aut}(X, \mu)$ (see e.g. [13, Corollary 15.2]). Given $f \in \mathrm{Aut}(X, \mu)$, the measure μ need not by f-ergodic. Write $\mathcal{E}(f)$ for the *ergodic decomposition* of μ with respect to f; precisely, $\mathcal{E}(f)$ is the measurable hull of the partition of (X, μ) into f-orbits.

Proposition 7.2. *Suppose there exists $f \in \mathrm{Aut}(X, \mu)$ with the following properties:*

(1) f preserves almost every H-orbit and commutes with the H-action: for a.e. $x \in X$ and every $h \in H$,

$$f(h \cdot x) = h \cdot f(x);$$

(2) $\mathcal{E}(f) \prec \mathcal{H}$.

Then, for μ-a.e. $x \in X$, the group $\mathrm{Isom}^H(H; [\nu_x^H])$ acts transitively on the support of ν_x^H.

Proof. By f-invariance of μ and the assumption that f preserves the canonical parametrizations of H-orbits, we have that $\nu^H_{f(x)} = \nu^H_x$ for almost every x; in particular, the measurable map $x \mapsto \nu^H_x$ is constant on f-ergodic components of μ.

The assumption that $\mathcal{E}(f) \prec \mathcal{H}$ implies for μ-a.e. x and ν^H_x-a.e. $h \in H$ that x and $h \cdot x$ are in the same f-ergodic component of μ; in particular for μ-a.e. x and ν^H_x-a.e. $h \in H$ we have

$$\nu^H_{h\cdot x} = \nu^H_x.$$

Since

$$(\Phi_x)_* \nu^H_x \propto (\Phi_{h\cdot x})_* \nu^H_{h\cdot x}$$

and since $\Phi^{-1}_{h\cdot x} \circ \Phi_x$ corresponds to right-translation by h^{-1} and we have that h^{-1}, and hence h, are elements of the group $\mathrm{Isom}^H(H; [\nu^H_x])$. Since $\mathrm{Isom}^H(H; [\nu^H_x])$ is a closed subgroup of H, its orbit in H is closed and the conclusion follows. □

Given an automorphism φ of H, write $d\varphi$ for the corresponding automorphism of the Lie algebra $\mathfrak{h}$. Assuming there exists a transformation $g \in \mathrm{Aut}(X, \mu)$ that preserves H-orbits and acts on H-orbits by a contracting automorphism φ, we obtain stronger properties of the groups $\mathrm{Isom}^H(H; [\nu^H_x])$.

Proposition 7.3. *Suppose there exists $g \in \mathrm{Aut}(X, \mu)$ and an automorphism $\varphi \in \mathrm{Aut}(H)$ with $\| d\varphi \| < 1$ such that for a.e. $x \in X$ and every $h \in H$,*

$$g(h \cdot x) = \varphi(h) \cdot g(x).$$

Moreover, suppose for μ-a.e. $x \in X$ that $\mathrm{Isom}^H(H; [\nu^H_x])$ acts transitively on the support of ν^H_x.

Then for μ-a.e. $x \in X$

(1) $\mathrm{Isom}^H(H; [\nu^H_x]) = \mathrm{Isom}^H(H; \nu^H_x)$*, and*
(2) ν^H_x *coincides (up to a choice of normalization) with the left-Haar measure on a connected Lie subgroup $L_x \subset H$.*

Proof. For (1), given $x \in X$ and $h \in \mathrm{Isom}^H(H; [\nu^H_x])$ set

$$c_x(h) = \frac{\nu^H_x(B^H(1) \cdot h)}{\nu^H_x(B^H(1))}.$$

For any $E \subset H$ with $\nu^H_x(E) > 0$ we then have

$$\nu^H_x(E \cdot h) = c_x(h) \nu^H_x(E).$$

We check the following hold for almost every x:

(a) For $h_1, h_2 \in \mathrm{Isom}^H(H; [\nu^H_x])$ we have

$$c_x(h_1 h_2) = c_x(h_2) c_x(h_1)$$

and hence obtain a homomorphism $c_x \colon \mathrm{Isom}^H(H; [\nu^H_x]) \to (\mathbb{R}^+, \times)$.
(b) $c_x(h) = c_{g(x)}(\varphi(h))$.
(c) $c_x \colon \mathrm{Isom}^H(H; [\nu^H_x]) \to \mathbb{R}^+$ is continuous.

Indeed (a) and (b) follow from properties of leaf-wise measures and g-invariance of μ. For (c), we have that $c_x\colon H \to \mathbb{R}^+$,

$$c_x\colon h \mapsto \frac{\nu_x^H(E \cdot h)}{\nu_x^H(E)}.$$

is both lower- and upper-semicontinuous by considering $E \subset H$, respectively, open or closed.

Given $\delta > 0$ and $x \in X$, let ϵ_x be such that for all $h \in B^H(\epsilon_x)$,

$$|1 - c_x(h)| < \delta.$$

We have that $\epsilon_x > 0$ for almost every $x \in X$. Given $R > 0$ and any $\epsilon > 0$ we have that $\varphi^n(B^H(R)) \subset B^H(\epsilon)$ for all sufficiently large n. By Poincaré recurrence of orbits of g to sets on which $x \mapsto \epsilon_x$ is bounded from below and applying (b), we see for almost every $x \in X$ that

$$|1 - c_x(h)| < \delta$$

for all $h \in \mathrm{Isom}^H(H; [\nu_x^H])$. Taking $\delta \to 0$, conclusion (1) then follows.

For (2), given $x \in X$ set $L_x := \mathrm{Isom}^H(H; \nu_x^H)$. We have that L_x is a closed subgroup of H and hence a Lie group. We claim L_x is connected for a.e. x. Indeed, let L_x' denote the connected component of L_x through the identity. Let r_x denote the minimal distance from the identity to any L_x'-orbit not-containing the identity. If $L_x' \neq L_x$ then $r_x > 0$. On the other hand, from (b) above and the assumptions on φ, we have $r_{g(x)} < \kappa r_x$ for some $0 < \kappa < 1$. If $L_x' \neq L_x$ for a positive measure set of x, we would thus obtain a contradiction with Poincaré recurrence to sets on which $x \mapsto r_x$ is bounded from below.

We thus have that ν_x^H coincides with a right-Haar measure on $L_x \subset H$ for almost every $x \in X$. The assumption on φ ensures that H is nilpotent. It thus follows that L_x is nilpotent. As nilpotent groups are unimodular, ν_x^H also coincides with a left-Haar measure on $L_x \subset H$ a.s. □

8. Structure theory of G and stable classes of roots

8.1. **Cartan and Iwasawa decompositions.** Let $\mathfrak{g}$ denote the Lie algebra of G. Fix a Cartan involution θ of $\mathfrak{g}$ and write $\mathfrak{k}$ and $\mathfrak{p}$, respectively, for the $+1$ and -1 eigenspaces of θ. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Recall that $\dim_{\mathbb{R}}(\mathfrak{a})$ is the $\mathbb{R}$-*rank* of G.

We let Σ denote the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$; elements of Σ are real linear functionals on $\mathfrak{a}$. A *base* (or a collection of *simple roots*) for Σ is a subset $\Pi \subset \Sigma$ that is a basis for the vector space $\mathfrak{a}^*$ and such that every non-zero root $\beta \in \Sigma$ is either a positive or a negative integer combination of elements of Π. For a choice of Π, elements $\alpha \in \Pi$ are called *simple* (positive) roots. Relative to a choice of base Π, let $\Sigma_+ \subset \Sigma$ be the collection of positive roots and let Σ_- be the corresponding set of negative roots.

For $\beta \in \Sigma$ write $\mathfrak{g}^\beta$ for the associated root space. Then $\mathfrak{n} = \bigoplus_{\beta \in \Sigma_+} \mathfrak{g}^\beta$ is a nilpotent subalgebra. We have that $\theta(\mathfrak{n}) = \bigoplus_{\beta \in \Sigma_-} \mathfrak{g}^\beta$. Write $\mathfrak{n}^- = \theta(\mathfrak{n})$.

Let A, N, and K be the analytic subgroups of G corresponding to $\mathfrak{a}, \mathfrak{n}$ and $\mathfrak{k}$. We also write $N^- = \theta(N)$ for the analytic subgroup corresponding to $\mathfrak{n}^-$. Then $G = KAN$ is the corresponding *Iwasawa decomposition* of G. When G has finite center, K is compact. Note that the Lie exponential $\exp\colon \mathfrak{g} \to G$ restricts to diffeomorphisms between $\mathfrak{a}$ and A and $\mathfrak{n}$ and N. Write $M = C_K(\mathfrak{a})$ for the centralizer of $\mathfrak{a}$ in K. Then $P = MAN$ is the *standard minimal parabolic subgroup*. We have that P is amenable as it is a compact extension of a solvable Lie group.

8.2. **Stable collections of roots.** We have the following definition.

Definition 8.1 (Stable collection of roots)**.** A collection of roots $C \subset \Sigma$ is called *stable* if there exist $s_1, \dots s_k \in \mathfrak{a}$ such that

$$C = \big\{\beta \in \Sigma : \beta(s_i) < 0 \text{ for all } 1 \leqslant i \leqslant k\big\}.$$

If $C \subset \Sigma$ is a stable collection of roots then the vector subspace

$$\mathfrak{u}^C := \bigoplus_{\beta \in C} \mathfrak{g}^\beta$$

is a nilpotent Lie subalgebra of $\mathfrak{g}$. Write U^C for the corresponding analytic subgroup of G. A maximal stable collection of roots $C \subset \Sigma$ corresponds to a choice of ordering of Σ and corresponding collection of negative roots. A minimal stable collection of roots corresponds to a positive proportionality class of roots in Σ which we refer to as a *coarse root* and typically denote by $\chi \subset \Sigma$. By the structure of abstract root systems, we have that every coarse root $\chi \subset \Sigma$ is either a singleton $\chi = \{\beta\}$ or is of the form $\chi = \{\beta, 2\beta\}$.

As a primary example, we consider the following construction.

Example 8.2 (Stable collection of roots transverse to a parabolic subgroup)**.** Let $G = KAN$ be an Iwasawa decomposition, $P = MAN$ a minimal parabolic subgroup, and $Q \supset P$ a parabolic subgroup. Let $\mathfrak{m}, \mathfrak{a}, \mathfrak{n}$ and $\mathfrak{q}$ denote, respectively, the Lie algebras of M, A, N and Q. Let Π be a base of simple positive roots so that $\mathfrak{n}$ is spanned by roots spaces corresponding to positive roots. Let $\mathfrak{n}^- = \theta(\mathfrak{n})$ be the Lie subalgebra spanned by roots spaces corresponding to negative roots relative to this ordering. Let $\mathfrak{q}^- = \mathfrak{q} \cap \mathfrak{n}^-$.

We claim there exists a stable collection of roots $C \subset \Sigma$ such that $\mathfrak{q}^-$ and $\mathfrak{u}^C$ are transverse and of complementary dimension in $\mathfrak{n}^-$ so that $\mathfrak{n}^- = \mathfrak{q}^- \oplus \mathfrak{u}^C$. Indeed by the structure of parabolic subalgebras (see for instance [14]) we have that

$$\mathfrak{q} := \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\beta \in \Sigma_+} \mathfrak{g}^\beta \oplus \bigoplus_{\beta \in \mathbb{Z}_{\leqslant 0}\text{-span}(\Delta)} \mathfrak{g}^\beta$$

for some subset $\Delta \subset \Pi$. The last direct sum is taken over all roots β that are non-positive integer combinations of elements of Δ.

Take C to be the collection of all negative roots

$$\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$$

(so c_α is a non-positive integer for every $\alpha \in \Pi$) such that $c_\alpha \neq 0$ for some $\alpha \in \Pi \smallsetminus \Delta$. We may find $s \in \mathfrak{a}$ such that

(1) $\alpha(s) = 0$ for $\alpha \in \Delta$;
(2) $\alpha(s) > 0$ for $\alpha \in \Pi \smallsetminus \Delta$.

It follows that $\beta(s) < 0$ for all $\beta \in C$ and that $\beta(s) \geqslant 0$ for all $\beta \in \Sigma \smallsetminus C$; in particular, C is a stable collection of roots.

Remark 8.3. Fix a norm on $\mathfrak{g}$. Let C be a stable collection of roots. Fix $s \in \mathfrak{a}$ with $\beta(s) < 0$ for all $\beta \in C$. Let $\mathfrak{u}^C$ be the unipotent subalgebra associated with C and let $U := U^C = \exp(\mathfrak{u}^C)$ be the corresponding subgroup of G. Then $\| \operatorname{Ad}(\exp(s))|_{\mathfrak{u}^C} \| < \kappa$ for some $0 < \kappa < 1$.

Let G act continuously on a metric space X. Let $f\colon X \to X$ be $f(x) = \exp(s)\cdot x$. Given $x \in X$ and $W \in \mathfrak{u}^C$ we have

$$f(\exp(W)\cdot x) = \exp(\operatorname{Ad}(\exp(s))(W)) \cdot f(x).$$

In particular, we have

$$f^n(\exp(W)\cdot x) = \exp(W_n)\cdot (f^n(x))$$

where $W_n = \exp(\operatorname{Ad}(\exp(ns))(W))$ so $\| W_n \| \leqslant \kappa^n \| W \|$ for all $n \geqslant 0$. Assuming the metric on U^C-orbits induced by $\| \cdot \|$ is compatible with the ambient metric on X, it follows that the U^C-orbit of x is contained in the stable set of x for the action of f.

8.3. **Structure of leaf-wise measures for stable collections of roots.** Let X be a complete, second countable, locally compact metric space and let G act continuously on X. Fix an Iwasawa decomposition $G = KAN$ of G and let μ be an ergodic, A-invariant Borel probability measure on X.

Given a stable collection of roots $C \subset \Sigma$, let

$$x \mapsto \nu_x^C := \nu_x^{U^C}$$

denote the family of leaf-wise measure on U^C associated to μ. We have the following proposition which is a simplification of the "product structure" of leaf-wise measures established by Einsiedler and Katok (c.f. [5, Proposition 5.1, Corollary 5.2] and [6, Theorem 8.4]). Roughly, we have that if ν_x^C has non-trivial support then ν_x^χ has non-trivial support for some coarse root $\chi \subset C$.

Proposition 8.4. *Suppose there is a stable collection of roots $C \subset \Sigma$ such that for almost every $x \in X$, the leaf-wise measure ν_x^C on U^C associated to μ is not a point-mass supported at the identity.*

Then there exists a coarse root $\chi \subset C$ such that for almost every $x \in X$, the leaf-wise measure ν_x^χ on U^χ associated to μ is not a point-mass supported at the identity.

Proof. Given any stable collection of roots $C \subset \Sigma$ we may find a coarse root $\chi \subset C$ and $s \in \mathfrak{a}$ such that $\beta(s) = 0$ for all $\beta \in \chi$ and

$$\beta'(s) < 0$$

for all $\beta' \in C \smallsetminus \chi$.

Let $C' = C \smallsetminus \chi$. Then C' is a stable collection of roots. If $\nu_x^C = \nu_x^{C'}$ for almost every x we may replace C with C' and proceed by induction on the cardinality of C. Thus, without loss of generality, we may assume there is $E \subset X$ with $\mu(E) > 0$ such that ν_x^C is not supported on $U^{C'}$ for $x \in E$. Since we assume μ is A-invariant and ergodic and since A normalizes $U^C, U^{C'}$, and U^χ, it follows that ν_x^C is not supported on $U^{C'}$ for almost every $x \in X$.

Let $f\colon X \to X$ be the map $f(x) = \exp(s) \cdot x$. Since $\chi(s) = 0$, $\exp(s)$ commutes with elements of U^χ. Since μ is f-invariant, we then have $\nu_x^\chi = \nu_{f(x)}^\chi$ for μ-a.e. x.

Since the assignment $x \mapsto \nu_x^\chi$ is measurable, by Lusin's theorem, given $\epsilon > 0$ there is a compact $K_\epsilon \subset X$ with $\mu(K_\epsilon) > 1 - \epsilon$ on which the map $x \mapsto \nu_x^\chi$ is continuous. For almost every $x \in X$ and ν_x^C-a.e. $u \in U^C \smallsetminus U^{C'}$ we have the following:

(1) The measure ν_x^C is not supported on $U^{C'}$.
(2) The identity element of U^χ is contained in the support of ν_x^χ and $\nu_{u\cdot x}^\chi$.
(3) There exist arbitrarily small values $\epsilon > 0$, such that $x \in K_\epsilon$ and $u \cdot x \in K_\epsilon$.
(4) For every $n \in \mathbb{N}$, we have $\nu_x^\chi = \nu_{f^n(x)}^\chi$ and $\nu_{u\cdot x}^\chi = \nu_{f^n(u\cdot x)}^\chi$.
(5) Taking $\epsilon < \frac{1}{2}$ sufficiently small, there exists $n_j \to \infty$ such that $f^{n_j}(x) \in K_\epsilon$ and $f^{n_j}(u \cdot x) \in K_\epsilon$.
(6) $u = hv$ where $h \in U^{C'}$ and $v \in U^\chi$; moreover for every $n \geqslant 0$, $f^n(u \cdot x) = u_n f^n(x)$ where $u_n = h_n v$ and $h_n \to 0$ as $n \to \infty$.

Taking $n_j \to \infty$ and passing to further subsequences, we may assume $f^{n_j}(x)$ and $f^{n_j}(u\cdot x)$ converge, respectively, to some $x_\infty \in K_\epsilon$ and $y_\infty \in K_\epsilon$. We have $y_\infty = v\cdot x_\infty$. We have that $\nu_{x_\infty}^\chi = \nu_x^\chi$ and $\nu_{y_\infty}^\chi = \nu_{u\cdot x}^\chi$; in particular, the identity element in U^χ is contained in the supports of $\nu_{x_\infty}^\chi$ and $\nu_{y_\infty}^\chi$. However, since $y_\infty = v\cdot x_\infty$ it follows that $\nu_{x_\infty}^\chi$ and hence ν_x^χ is supported at $v \in U^\chi$ and hence ν_x^χ is not an atom supported at the identity. □

9. Proof of Proposition 5.3

We use the results and constructions from Sections 6–8 to prove Proposition 5.3. From the discussion in Section 5, this completes the proof of Theorem 1.2.

Proof of Proposition 5.3. Let ν be the Q-invariant measure on X^Γ constructed in Section 5. By construction, the projection $(X^\Gamma, \nu) \to (G/\Gamma, \lambda_{G/\Gamma})$ is essentially injective; in particular, the projection $(X^\Gamma, \nu) \to (G/\Gamma, \lambda_{G/\Gamma})$ is a measurable isomorphism. It is well-known that $\lambda_{G/\Gamma}$ is ergodic under the action of any non-compact subgroup $H \subset G$. In particular, it follows for any noncompact subgroup $H \subset Q$, that the action of H on (X^Γ, ν) is ergodic.

Suppose for the sake of contradiction that there exists a positive ν-measure subset of $x \in X^\Gamma$ for which the leaf-wise measure $\nu_x^{N^-}$ is not supported on Q^-. Since A preserves ν and acts ergodically on (X^Γ, ν) and since Q^- is normalized by A, it follows that $\nu_x^{N^-}$ is not supported on Q^- for ν-almost every $x \in X^\Gamma$.

Let C be the stable collection of roots as in Example 8.2 with $\mathfrak{u}^C$ transverse to $\mathfrak{q}^-$ in $\mathfrak{n}^-$. Then the subgroup $U^C \subset N^-$ is transverse to Q^--orbits in N^-. Since $\nu_x^{N^-}$ is Q^--invariant for ν-almost every x, it follows that $\nu_x^{U^C}$ is not supported at the identity for almost every x.

By Proposition 8.4, we may find a coarse root $\chi \in C$ such that ν_x^χ is not supported at the identity for a positive measure subset of $x \in X^\Gamma$; by ergodicity, this then holds for ν-almost every $x \in X^\Gamma$.

Fix $s \in \mathfrak{a} \smallsetminus \{0\}$ with $\chi(s) = 0$ and $s' \in \mathfrak{a} \smallsetminus \{0\}$ with $\chi(s') < 0$. Set $f, g \in \mathrm{Aut}(X^\Gamma, \nu)$ to be

$$f(x) = \exp(s) \cdot x, \qquad g(x) = \exp(s') \cdot x.$$

We have that f is an ergodic transformation of (X^Γ, ν). By Proposition 7.2 and Proposition 7.3 (with the above f and g), for ν-a.e. $x \in X^\Gamma$ the measure ν_x^χ is the left-Haar measure on a connected Lie subgroup $L_x \subset U^\chi$. Since ν_x^χ is not supported at the identity, L_x has positive dimension. Moreover, since f intertwines the parametrizations of H-orbits and preserves the measure μ, the Lie algebras of L_x and $L_{f(x)}$ coincide whence the map $x \mapsto L_x$ is f-invariant. By f-ergodicity of ν, there is a subgroup $L \subset U^\chi$ such that $L_x = L$ for ν-almost every x. It follows from Claim 6.5 that μ is L-invariant. But L is not a subgroup of Q, contradicting the choice of Q as the (maximal) stabilizer of ν. □

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