Chapter 2

Lattices and nilmanifolds

In this chapter, we discuss criteria for a discrete subgroup Γ in a simply connected nilpotent Lie group G to be a lattice, as well as properties of the resulting quotient G/Γ.

2.1 Haar measure of a nilpotent Lie group

The map exp is a diffeomorphism between the Lie algebra $\mathfrak{g}$ of a simply connected Lie algebra G and G itself. On both $\mathfrak{g}$ and G there are natural volume forms. On $\mathfrak{g}$, this is just the Euclidean volume denoted by $\mathbf{m}_{\mathfrak{g}}$. On G, there is a natural left invariant measure, the left Haar measure, denoted by $\mathbf{m}_G$. The choices of $\mathbf{m}_{\mathfrak{g}}$ and $\mathbf{m}_G$ are unique up to a renormalizing factor.

Proposition 2.1.1. Suppose G is a simply connected nilpotent Lie group and $\mathfrak{g}$ is its Lie algebra. After renormalizing if necessary, the pushforward measure $\exp_* \mathbf{m}_{\mathfrak{g}}$ coincides with $\mathbf{m}_G$.

Proof. Fix a filtration $\{\mathfrak{g}_i\}_{i=1}^r$ (for example the central lower series $\{\mathfrak{g}_{(i)}\}$) and a Mal'cev basis $\mathcal{X}$ adapted to it. Since the left Haar measure is unique to renormalization, it suffices to show that $\exp_* \mathbf{m}_{\mathfrak{g}}$ is invariant under left multiplication. This is equivalent to that $\mathbf{m}_g$ is invariant under left multiplications in the group structure $(\mathfrak{g}, \odot)$.

On $\mathfrak{g}$, use the linear coordinates (1.16) determined by $\mathcal{X}$. Then for $U = \sum_{j=1}^m u_j X_j$ and $V = \sum_{j=1}^m V_j X_j$, $W = U \odot V$ is given by the formula (1.18). Thus the partial derivative in V of $U \odot V$ can be written in block form as

$$\begin{pmatrix} \mathrm{Id} & * & \cdots & * \\ & \mathrm{Id} & & * \\ & & \ddots & \vdots \\ & & & \mathrm{Id} \end{pmatrix},$$

where the i-th block correspond to the $\mathbf{v}_i$ component. Since the determinant of this matrix is 1, the map $V \to U \odot V$ preserves the Euclidean volume $\mathbf{m}_{\mathfrak{g}}$. This completes the proof. □

Observe that the partial derviative in U of $U \odot V$ has the same block form as above, hence $\mathbf{m}_{\mathfrak{g}}$ is also right invariant in $(\mathfrak{g}, \odot)$. So we get:

Corollary 2.1.2. *Every nilpotent Lie group G is unimodular, i.e. its left and right Haar measures coincide up to rescaling.*

Proof. If G is simply connected, the remark above asserts that $\mathbf{m}_G = \exp_* \mathbf{m}_{\mathfrak{g}}$ is both left invariant and right invariant. This yields the unimodularity as desired.

For a connected nilpotent Lie group G, its universal cover $\widetilde{G}$ is a Lie group with the same Lie algebra $\mathfrak{g}$ and hence, by Theorem 1.5.7, is nilpotent as well. The left and right Haar measures of $\widetilde{G}$ coincide and descend to the corresponding Haar measures on G, which still coincide (up to rescaling).

For a disconnected nilpotent Lie group G, let G^0 be its identity component, which is still nilpotent and is thus unimodular. It suffices to note in the case the product measure between the counting measure on G/G^0 and $\mathbf{m}_{G^0}$ is both left and right invariant on G. □

Exercises

Exercise 2.1.1. Describe the Haar measure of the group of upper triangular unipotent $d \times d$ matrices.

Exercise 2.1.2. Show that for $d \geq 2$, the group of upper triangular $d \times d$ matrices with positive diagonal entries is not unimodular.

2.2 Lattices

Definition 2.2.1. *Given a connected Lie group G, a discrete subgroup $\Gamma \subseteq G$ is called a* **lattice** *if the quotient G/Γ admits a finite measure that is left invariant by elements of G.*

Lemma 2.2.2. *If G has a lattice, then G is unimodular.*

Proof. Because Γ is discrete, one can lift the left invariant meaure $\mathbf{m}_{G/\Gamma}$ to a left invariant measure $\mathbf{m}_G$ by making the projection $G \to G/\Gamma$ a locally volume preserving map. For every $h \in G$, $(R_h)_*\mathbf{m}_G$ is also left invariant, where R_h is the right multiplication by h. Because left Haar measure is unique up to rescaling, $(R_h)_*\mathbf{m}_G = \chi(h)\mathbf{m}_G$ for a function $\chi : G \to \mathbb{R}_{>0}$. Note that χ is a group morphism into the multiplicative group $\mathbb{R}_{>0}$. Since the lifting is equivariant under the deck transformation group Γ, $\Gamma \subseteq \ker \chi$. The pushforward μ of $\mathbf{m}_{G/\Gamma}$ from G/Γ to $\mathbb{R}_{>0}$ by $g \to \chi(g)$ is invariant under multiplication by $\chi(k)$ for every $k \in G$. Unless $\chi \equiv 1$, which means $\mathbf{m}_G$ is also right invariant, the measure μ on $\mathbb{R}_{>0}$ in invariant under multiplicationby λ for some $\lambda > 1$. Such a measure cannot have finite totall mass, though $\mu(\mathbb{R}_{>0}) = \mathbf{m}_{G/\Gamma}(G/\Gamma) < \infty$. This yields a contradiction. Hence $\mathbf{m}_G$ is right invariant and G is unimodular. ☐

Example 2.2.3. $\mathbb{Z}^n$ is a lattice of $\mathbb{R}^n$.

Hereafter, G will be assumed to be a simply connected nilpotent Lie group. We will identify G with its Lie algebra $\mathfrak{g}$ via the exponential map exp. Using this identification, the group structure can be thought of as $(\mathfrak{g}, \odot)$ where $\odot$ is from (1.11). We remark that:

Proposition 2.2.4. Every simply connected nilpotent Lie group G is an algebraic group.

That is, using the linear coordinates (1.16) of $\mathfrak{g}$, G is identified with the affine algebraic variety[1] $\mathfrak{g} \cong \mathbb{R}^m$ and the group operations, which are the multiplication $\odot$ and the inversion $X \to -X$, are polynomial.

Since every connected closed subgroup H is identified with its Lie algebra $\mathfrak{h}$, which is a subspace of $\mathfrak{g}$. It is clear that:

Lemma 2.2.5. *In Proposition 2.2.4, every connected closed subgroup is a Zariski closed algebraic subgroup.*

The main result of this section is:

Theorem 2.2.6. *Let G be a simply connected nilpotent Lie group endowed with the algeraic structure of its Lie algebra $\mathfrak{g}$, and $\Gamma \subset G$ be a discrete subgroup, then the following are equivalent:*

[1] Algebraic subvarieties of $\mathfrak{g}$ are the zero sets of finite arrays of polynomial equations. They are the closed subsets of a topology, the Zariski topology.

(1) Γ *is a lattice;*
(2) Γ *is Zariski dense;*
(3) Γ *is not contained in any proper connected closed subgroup of* G*;*
(4) $\exp^{-1}\Gamma$ *is not contained in any proper vector subspace of* $\mathfrak{g}$*.*
(5) Γ *is cocompact, i.e.* G/Γ *is compact.*

Proof. We shall show (5)⇒(1)⇒(2)⇒(4)⇒(3)⇒(2)⇒(5).

(5)⇒(1). Because G is unimodular by Corollary 2.1.2, there is a bi-invariant Haar measure $\mathbf{m}_G$, which projects to a measure $\mathbf{m}_{G/\Gamma}$ on G/Γ by right invariance. Moreover $\mathbf{m}_{G/\Gamma}$ inherits the left invariance property.

The compactness of G/Γ means that there is a compact set $\Omega \subset G$ such that $\bigcup_{\gamma\in\Gamma}\Omega\gamma = G$. Then $\mathbf{m}_{G/\Gamma}(G/\Gamma) \leq \mathbf{m}_G(\Omega) < \infty$. So Γ is a lattice.

(1)⇒(2). Let H be the Zariski closure of Γ, i.e. the smallest closed algerbaic subvariety of G containing Γ. Then as a standard fact about topological groups, H is a subgroup of G. (Exercise 2.2.1)

Let H^0 be the identity component of H in Zariski topology. Then H^0 is a subgroup of H and has finite index. Since every Zariski connected closed subvariety is connected and closed in the usual Hausdorff topology. H^0 is a connected closed subgroup of G. By Lemma 1.5.16, $H = H^0$. In other words, H is Zariski connected and connected. By Theorem 1.1.18, H is a Lie subgroup, whose Lie algebra is $\mathfrak{h} := \exp^{-1} H$. Assume for contradiction that $H \subsetneq G$, then $\mathfrak{h} \subsetneq \mathfrak{g}$.

Let $\mathfrak{g}_{(2)} = [\mathfrak{g},\mathfrak{g}]$ be the commutator subalgebra of $\mathfrak{g}$ and $G_{(2)} = \exp\mathfrak{g}_{(2)}$. Then $\mathfrak{g}_{(2)}$ is an ideal and $G_{(2)}$ is a connected closed normal subgroup of G. Because of normality, $HG_{(2)}$ is a subgroup of G, which is again connected and closed and hence a Lie subgroup. Its Lie subalgebra is $\mathfrak{h}+\mathfrak{g}_{(2)}$. By Lemma 1.4.12, $\mathfrak{h}+\mathfrak{g}_{(2)} \neq \mathfrak{g}$ as $\mathfrak{h}\neq\mathfrak{g}$. So $HG_{(2)} = \exp(\mathfrak{h}+\mathfrak{g}_{(2)})$ is a proper subgroup of G. Moreover, $\mathfrak{h}+\mathfrak{g}_{(2)}$ is an ideal of $\mathfrak{g}$, because

$$[\mathfrak{g},\mathfrak{h}+\mathfrak{g}_{(2)}] \subseteq [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}_{(2)} \subseteq \mathfrak{h}+\mathfrak{g}_{(2)}.$$

Thus $HG_{(2)}$ is a normal subgroup in G by Lemma 1.5.4.

Consider the quotient group $G/HG_{(2)}$, which is non-trivial by the paragraph below, abelian as it is a quotient of $G/G_{(2)}$, which is abelian, and simply connected by Corollary 1.5.14. Thus there exists $d \geq 1$ such that $G/HG_{(2)}$ is isomorphic to $\mathbb{R}^d$ as a group.

Take the pushforward measure $\mu = \tau_*\mathbf{m}_{G/\Gamma}$ on $G/(HG_{(2)}) \cong \mathbb{R}^d$ by the map $\tau : g \to gHG_{(2)}$. (Note that τ is well-defined on G/Γ since $\Gamma \subset HG_{(2)}$.) Then μ has finite volume and is invariant under left translations, which is impossible on $\mathbb{R}^d$. The desired contradiction is hence obtained.

(3)⇒(2). Again, let H be the Zariski closure of Γ. Then as above, H must be a connected closed subgroup of G. So by assumption, $H = G$, which means that Γ is Zariski dense in G.

(2)⇒(4). This implication is obvious because the Zariski topology of G is inherited from that of $\mathfrak{g}$ via exp, and a proper vector subspace is a proper algebraic subvariety.

(4)⇒(3). If $\Gamma \subseteq H$ for a proper connected closed subgroup H of G, then $\exp^{-1}\Gamma \subseteq \mathfrak{h}$, which is a proper vector subspace of $\mathfrak{g}$.

(2)⇒(5). We prove by induction in the step of nilpotency. Suppose G is 1-step nilpotent, i.e. an abelian Lie group. Then G is isomorphic to $\mathbb{R}^d$ for some d. Since Γ is a discrete subgroup, which (by the implication (2)⇒(3) above) is not contained in any proper vector subspace, it must be isomorphic to a discrete embedding of $\mathbb{Z}^d$ into $\mathbb{R}^d$. In this case, $G/\Gamma \cong \mathbb{T}^d$ is compact.

Now assume the implication is known for all steps of nilpotency up to $s-1$, where $s \geq 2$.

Recall that $G_{(2)} = \exp \mathfrak{g}_{(2)}$ is the closure (in the usual topology) $\overline{[G, G]}$ of the group generated by all commutator elements $ghg^{-1}h^{-1}$ by Corollary 1.5.11. The discrete subgroup $[\Gamma, \Gamma] \subseteq \Gamma \cap G_{(2)}$ generated by $\{\gamma\eta\gamma^{-1}\eta^{-1} : \gamma, \eta \in \Gamma\}$ is hence Zariski dense in $G_{(2)}$ (since the Zariski topology is weaker than the usual topology). Because $G_{(2)}$ has step of nilpotency $s-1$, by inductive hypothesis, $[\Gamma, \Gamma]$ is a cocompact lattice in $G_{(2)}$. In consequence, the intermediate group $\Gamma_{(2)} = \Gamma \cap G_{(2)}$ is also a cocompact lattice in $G_{(2)}$.

The quotient $\Gamma/\Gamma_{(2)}$ naturally sits in $G/G_{(2)}$ as a subgroup. We shall show that it is discrete in the induced topology of $G/G_{(2)}$. This is equivalent to the statement that if for a sequence $\gamma_n \in \Gamma$, $\gamma_n\Gamma_{(2)} \to e$ in $G/G_{(2)}$ then $\gamma_n \in G_{(2)}$ for sufficiently large n.

The convergence to identity in $G/G_{(2)}$ means that there is a sequence $\epsilon_n \to e$ in G, such that $\gamma_n \in \epsilon_n G_{(2)}$, i.e. $\exists g_n \in G_{(2)}$ such that $\gamma_n = \epsilon_n g_n$. By cocompactness of $\Gamma_{(2)}$, g_n can be written as $h_n\beta_n$ where $\beta_n \in \Gamma_{(2)}$ and h_n belongs to a given compact subset $\Omega \subset G_{(2)}$. Thus $\gamma_n = \epsilon_n h_n \beta_n$.

Because ϵ_n and h_n are precompact sequences, $\gamma_n\beta_n^{-1} = \epsilon_n h_n$ fall into a given compact set $\Omega' \in G$ for all n. On the other hand, $\gamma_n\beta_n^{-1}$ is from the discrete subgroup Γ. In particular, there are only finitely many possible values of $\gamma_n\beta_n^{-1} \in \Omega' \cap \Gamma$. So $\gamma_n\Gamma_{(2)} = \gamma_n\beta_n^{-1}\Gamma_{(2)}$ can only assume finitely many possible values. Thus the convergence of this sequence to the identity in $G/G_{(2)}$ forces it to eventually assume the identity value. This proves the discreteness of $\Gamma/\Gamma_{(2)}$.

Remark now that $G/G_{(2)}$ is abelian and also simply connected by Corollary 1.5.14. So $G/G_{(2)} \cong \mathbb{R}^d$ for some d. Moreover, $\Gamma/\Gamma_{(2)}$ is a discrete subgroup that does not belong to any proper connected closed subgroup $H \subset G/G_{(2)}$, since otherwise Γ would belong to the preimage of H, which is a proper connected closed subgroup of G. By the $s = 1$ case, $\Gamma/\Gamma_{(2)}$ is cocompact in $G/G_{(2)}$.

We conclude the proof by noting that G/Γ is a continuous fiber bundle over $(G/G_{(2)})/(\Gamma/\Gamma_{(2)})$ whose fibers are isomorphic to $G_{(2)}/\Gamma_{(2)}$. It is compact because both the base and the fibers are compact. □

Example 2.2.7. Consider the $2n + 1$ Heisenberg Lie group H_{2n+1} from Example 1.5.13. The discrete subset $\Gamma = \{(\mathbf{x}, \mathbf{y}, z) : \mathbf{x}, \mathbf{y} \in \mathbb{Z}^n, z \in \mathbb{Z}\}$ is closed under the group operations and hence a discrete subgroup. As an algebraic group, H_{2n+1} has the Zariski topology of the underlying $\mathbb{R}^{2n+1}$ (see Exercise 1.5.2). Moreover, $\Gamma = \mathbb{Z}^{2n+1}$ is Zariski dense. (this fact comes from application of Theorem 2.2.6 to the abelian Lie group $\mathbb{R}^{2n+1}$. By Theorem 2.2.6, Γ is a lattice.

From Theorem 2.2.6 we can deduce the following more detailed characterization of lattices.

Corollary 2.2.8. *In the setting of the Theorem 2.2.6, for all index pairs $1 \leq j \leq i \leq s$, $\Gamma \cap G_{(i)}$ and $(\Gamma \cap G_{(j)})/(\Gamma \cap G_{(i)})$ are respectively lattices in $G_{(i)}$ and $G_{(j)}/G_{(i)}$.*

Here s is the step of nilpotency.

Proof. Similar to the proof of the implication (2)⇒(5) of 2.2.6, one can show that $\Gamma_{(i)} = \Gamma \cap G_{(i)}$ is Zariski dense in $G_{(i)}$, and thus a cocompact lattice by the theorem.

If $j < i$, then $\Gamma_{(j)}/\Gamma_{(i)}$ is naturally identified with a subgroup of $G_{(j)}/G_{(i)}$. Again, the same proof as for the implication (2)⇒(5) of the theorem applies here and shows this subgroup is discrete. It is then a lattice because property (3) of Theorem 2.2.6 passes from $\Gamma_{(j)}$ to $\Gamma_{(j)}/\Gamma_{(i)}$. □

Exercises

Exercise 2.2.1. Let G be the group

$$\left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\} \times \left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\},$$

prove that the subset

$$\left\{\left(\begin{pmatrix} 1 & x_1 + x_2\sqrt{2} & z_1 + z_2\sqrt{2} \\ & 1 & y_1 + y_2\sqrt{2} \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_1 - x_2\sqrt{2} & z_1 - z_2\sqrt{2} \\ & 1 & y_1 - y_2\sqrt{2} \\ & & 1 \end{pmatrix}\right)\right.$$
$$\left. : x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{Z}\right\}$$

is a lattice.

2.3 Rationality of lattices

The algebraic group structure of a simply connected Lie group G is prescribed by the structural constants with respect to a Mal'cev basis $\mathcal{X}$. The existence of a lattice has consequences about these constants.

Theorem 2.3.1. *A simply connected nilpotent Lie group G admits a lattice Γ if and only if its Lie algebra $\mathfrak{g}$ has a Mal'cev basis $\mathcal{X}$ (adapted to some filtration) with respect to which the structural constants are in $\mathbb{Q}$.*

Moreover, in this case the basis $\mathcal{X}$ can be chosen such that the preimage $\exp^{-1}\Gamma$ consists of rational vectors in the coordinate system given by $\mathcal{X}$.

Proof of the "only if" part. Assume Γ is a lattice. We use the lower central series filtration $\{\mathfrak{g}_{(i)}\}_{i=1}^{s+1}$ and set $G_{(i)} = \exp \mathfrak{g}_{(i)}$, $\Gamma_{(i)} = \Gamma \cap G_{(i)}$.

By Corollary 2.2.8, $\Gamma_{(i)}/\Gamma_{(i+1)}$ is a lattice in $G_{(i)}/G_{(i+1)}$ for every index i. Since the later is an abelian Lie group, it identifies with its Lie algebra $\mathfrak{g}_{(i)}/\mathfrak{g}_{(i+1)}$ as a group (with exp being the identity map). $\Gamma_{(i)}/\Gamma_{(i+1)}$ can be regarded as a lattice in the vector space $\mathfrak{g}_{(i)}/\mathfrak{g}_{(i+1)} \cong \mathbb{R}^{d_i}$, and we can choose a $\mathbb{Z}$-basis $V_{m-m_i+1}, \cdots, V_{m-m_{i+1}}$ of this lattice. Here d_i, m, and m_i are defined as in §2.1.

For all $m - m_i + 1 \leq j \leq V_{m-m_{i+1}}$, one can choose $\gamma_j \in \Gamma_{(i)}$ such that $\gamma_j\Gamma_{(i+1)}$ is identified with V_j. Write $X_j = \exp^{-1}\gamma_j \in \mathfrak{g}_{(i)}$. Performing this for all the indices $1 \leq i \leq s$, we obtain a collection $\mathcal{X} = \{X_1, \cdots, X_m\}$. It will be shown that $\mathcal{X}$ is a Mal'cev basis with rational structural constants.

To show $\mathcal{X}$ is a Mal'cev basis, it suffices to know $X_{m-m_i+1}, \cdots, X_{m-m_{i+1}}$ project to a basis of $\mathfrak{g}_{(i)}/\mathfrak{g}_{(i+1)}$. Indeed, X_j projects to V_j thanks to the commutative diagram (1.14), and the claim follows. So it remains to show the rationality of the structural contants.

It will be shown by induction for all indices $1 \leq i \leq s$ that:

Claim 2.3.2. *(1) There exists a positive integer Q_i, such that for all $1 \leq \alpha \leq m$, $m - m_i + 1 \leq \beta \leq m$ and $m - m_{i+1} + 1 \leq j \leq m$, the structural constant $c_{\alpha\beta}^j$ in (1.15) is in $Q_i^{-1}\mathbb{Z}$.*

(2) $\Gamma_{(i)}$ is generated by $\{\gamma_j : m - m_i + 1 \leq j \leq m\}$. Indeed, every $\gamma \in \Gamma_{(i)}$ can be written as $\gamma_{m-m_j+1}^{r_{m-m_j+1}} \cdots \gamma_m^{r_m}$ for integers $r_{m-m_j+1}, \cdots, r_m$.

(3) In addition,

$$\exp^{-1} \Gamma_{(i)} \subseteq \bigoplus_{j=m-m_i+1}^{m} Q_i^{-1}\mathbb{Z}X_j. \tag{2.1}$$

The base case is $i = s$. In this case there are no structural constants to consider since $m_{s+1} = 0$. Hence part (1) of the claim is empty. In addition, $\Gamma_{(s)} = \Gamma_{(s)}/\Gamma_{(s+1)}$ is identified, via the exponential map, with the lattice generated by $X_{m-m_s+1} = V_{m-m_s+1}, \cdots$, and $X_m = V_m$ in $\mathfrak{g}_{(s)} = \mathfrak{g}_{(s)}/\mathfrak{g}_{(s+1)}$. So part (2) is true.

We now assume that the claim is known for index $i+1$. Then the vector $Y = \exp^{-1}(\gamma_\alpha \gamma_\beta \gamma_\alpha^{-1} \gamma_\beta^{-1})$ is in $\bigoplus_{\nu=m-m_{i+1}+1}^{m} Q_{i+1}^{-1}\mathbb{Z}X_\nu$.

By Corollary 1.1.13, $\operatorname{Ad}_{\gamma_\alpha} X_\beta = \exp^{-1}(\gamma_\alpha \gamma_\beta \gamma_\alpha^{-1})$ and thus equals $Y \odot X_\beta$. In particular, $\operatorname{Ad}_{\gamma_\alpha} X_\beta - Y - X_\beta$ is a finite linear combination, with coefficients from $P^{-1}\mathbb{Z}$, of repeated Lie brackets that only involve Y and X_β. Here P is a positive integer depending only on the step of nilpotency s. Note that each repeated Lie bracket can have at most $s - i$ layers.

Because of inductive hypothesis, one sees that every intermediate Lie bracket, and hence eventually every term in the linear combination, is in $\bigoplus_{j=m-m_{i+2}+1}^{m} Q_{i+1}^{-(s-i+1)}\mathbb{Z}X_j$. Hence, after taking coefficients into account and adding Y, we obtain that

$$(\operatorname{Ad}_{\gamma_\alpha} - \operatorname{Id})X_\beta \in \bigoplus_{j=m-m_{i+1}+1}^{m} P^{-1}Q_{i+1}^{-(s-i+1)}\mathbb{Z}X_j. \tag{2.2}$$

Next, consider the Lie bracket

$$[X_\alpha, X_\beta] = \operatorname{ad}_{X_\alpha} X_\beta = (\exp^{-1}(\operatorname{Ad}_{\exp X_\alpha}))(X_\beta) = (\exp^{-1}(\operatorname{Ad}_{\gamma_\alpha}))(X_\beta).$$

Here $\operatorname{Ad}_{\gamma_\alpha}$ is regarded as an element of the linear group $\operatorname{GL}(\mathfrak{g})$, and $\exp^{-1}(\operatorname{Ad}_{\gamma_\alpha})$ is the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(\operatorname{Ad}_{\gamma_\alpha} - \operatorname{Id})^n$. (This requires some justification, see Exercise 1.4.5.) Recall that $(\operatorname{Ad}_{\gamma_\alpha} - \operatorname{Id})^s = 0$ by Corollary 1.5.15. Hence the series at hand becomes a finite sum and ends at index $n = s - 1$.

Because (2.2) hold for all $1 \leq \alpha' \leq m$ and $m - m_i + 1 \leq \beta' \leq m$, it follows by iteration of this fact that

$$(\mathrm{Ad}_{\gamma_\alpha} - \mathrm{Id})^n X_\beta \in \bigoplus_{j=m-m_{i+1}+1}^{m} (P^{-1} Q_{i+1}^{-(s-i+1)})^{s-1} \mathbb{Z} X_j$$

for every $n \leq s-1$. So we conclude that

$$[X_\alpha, X_\beta] \in \bigoplus_{j=m-m_{i+1}+1}^{m} ((s-1)!)^{-1} (P^{-1} Q_{i+1}^{-(s-i+1)})^{s-1} \mathbb{Z} X_j.$$

This proves part (1) in Claim 2.3.2 for level i with $Q_i = (s-1)! P^{s-i} Q_{i+1}^{s-i}$.

On the other hand, suppose $\eta \in \Gamma_{(i)}$ then $\eta \Gamma_{(i+1)}$ can be viewed as an element of the lattice generated by $V_{m-m_i+1}, \cdots V_{m-m_{i+1}}$. Recall for each of these V_j is identified with the corresponding equivalence class $\gamma_j \Gamma_{(i+1)} = (\exp X_j)\Gamma_{(i+1)}$. Therefore, one can find integers r_j for each $m - m_i + 1 \leq j \leq m - m_{i+1}$ such that

$$\eta \in \exp(r_{m-m_i+1} X_{m-m_i+1}) \cdots \exp(r_{m-m_{i+1}} X_{m-m_{i+1}}) \Gamma_{(i+1)}.$$

By induction, part (2) of the claim is established.

Moreover, by part (3) of the claim with index $i+1$, there is $Z \in \bigoplus_{j=m-m_{i+1}+1}^{m} Q_{i+1}^{-1} \mathbb{Z} X_j$ such that

$$\eta = \exp(r_{m-m_i+1} X_{m-m_i+1}) \cdots \exp(r_{m-m_{i+1}} X_{m-m_{i+1}}) \exp Z.$$

Applying Baker-Campbell-Hausdorff formula, one see that $\exp^{-1} \eta$ is a rational linear combination, with coefficients from $P^{-(s-i)}\mathbb{Z}$, of X_{m-m_i+1}, $\cdots$, $X_{m-m_{i+1}}$, Z and iterated Lie brackets among them. Here every iterated Lie bracket has at most $s-i$ layers, and $P = P(s)$ is the same positive integer as before.

Since $Z \in \bigoplus_{j=m-m_{i+1}+1}^{m} Q_{i+1}^{-1} \mathbb{Z} X_j$ and $Q_i | Q_{i+1}$, it follows from part (1) that $\exp^{-1} \eta \in \bigoplus_{j=m-m_i+1}^{m} P^{-(s-i)} Q_i^{-(s-i+1)} \mathbb{Z} X_j$. This inductively proves part (3), after redenoting $P^{(s-i)} Q_i^{(s-i+1)}$ by Q_i.

In summary, we have established inductively Claim 2.3.2.

In particular, the "only if" part of Theorem 2.3.1 is covered by the $i = 1$ case. ∎

Proof of the "if" part. Assume now with respect to a filtration $\{\mathfrak{g}_i\}_{i=1}^{r+1}$ and a Mal'cev basis $\mathcal{X} = \{X_1, \cdots, X_m\}$ adapted to it, the structural constants are rational. Since there are only finitely many structural constants, there is $Q \in \mathbb{N}$ that every structural constant $c^j_{\alpha\beta}$ is in $Q^{-1}\mathbb{Z}$.

Let s be the step of nilpotency of G. Then the Baker-Campbell-Hausdorff formula is a rational linear combination of finitely many terms, each of which has at most s layers of Lie brackets. By modifying Q, we may assume that all coefficients are also in $Q^{-1}\mathbb{Z}$.

Let $\Lambda = \bigoplus_{j=1}^m Q^{r+1}\mathbb{Z}X_j$. We claim $\Gamma = \exp\Lambda$ is a lattice of G.

We first need to confirm that Γ is a subgroup, or equivalently Λ is closed under the group multiplication $\odot$ and the inversion $X \to -X$ on $\mathfrak{g}$. The inversion part is obvious. For the multiplication, let $X, Y \in \Lambda$. Then $X \odot Y$ is a finite sum of terms from $Q^{2(r+1)}\mathbb{Z}(X_\alpha \odot X_\beta)$. It suffices to prove each such term is in Λ.

Indeed, $X_\alpha \odot X_\beta$ us a finite sum of terms in the form

$$b[Y_1, [Y_2, \cdots, [Y_t, Y_{t+1}] \cdots]]$$

where $0 \leq t \leq r$ and $b \in Q^{-1}\mathbb{Z}$, and every Y_j is either X_α or X_β. it suffices to show that each of such terms is in $Q^{-2(r+1)}\Lambda = \bigoplus_{j=1}^m Q^{-(r+1)}\mathbb{Z}X_j$. By iteration of (1.15) over at most t levels of brackets, the repeated Lie bracket above is in $\sum_{j=1}^m Q^{-t}X_j \subseteq \bigoplus_{j=1}^m Q^{-r}\mathbb{Z}X_j$. After multiplying by $b \in Q^{-1}\mathbb{Z}$, every term is in $\bigoplus_{j=1}^m Q^{-(r+1)}\mathbb{Z}X_j$. We conclude that Γ is a discrete subgroup of G.

Furthermore Γ is Zariski dense of G because the Zariski topology of G is given that of $\mathfrak{g}$ via the exponential map. In these coordinates Γ is represented by Λ, which is a cocompact lattice of the vector space $\mathfrak{g}$ (with respect to the group rule $+$ instead of $\odot$) and hence Zariski dense by Theorem 2.2.6. So by another direction of Theorem 2.2.6, Γ is a lattice. □

Part (2) from Claim 2.3.2 can be reformulated as follows

Corollary 2.3.3. *Every lattice in a simply connected nilpotent Lie group is finitely generated. In fact, in Theorem 2.3.2 the Mal'cev basis $\mathcal{X}$ can be chosen so that Γ is generated by $\{\exp X_j : 1 \leq j \leq m\}$.*

Theorem 2.3.1 actually provides a coordinate system on G that is defined over $\mathbb{Q}$, such that all group operations are given by polynomials with rational coefficients. In terminologies of algerbaic geometry, a simply connected nilpotent Lie group that admits a lattice is an algebraic group defined over $\mathbb{Q}$. This in fact asserts that Γ uniquely determines a $\mathbb{Q}$-structure on the algebraic group G.allows us to define the rational points in G.

Definition 2.3.4. *Suppose* Γ *is a lattice in a simply connected nilpotent Lie group* G*. An element* $g \in G$ *(resp.* $X \in \mathfrak{g}$*) is* **rational** *if* $\exp^{-1} g$ *(resp.* X*) belongs to* $\bigoplus_{j=1}^m \mathbb{Q}X_j$*, where* $\mathcal{X} = \{X_1, \cdots, X_m\}$ *is a Mal'cev basis such that* $\exp^{-1}\Gamma \subseteq \bigoplus_{j=1}^m \mathbb{Q}X_j$*. We will denote respectively by* $G_{\mathbb{Q}}$ *and* $\mathfrak{g}_{\mathbb{Q}}$ *the rational elements of* G *and* $\mathfrak{g}$*.*

A legitimate question is whether the definition above depends on the choice of $\mathcal{X}$. The answer is no, as demonstrated by the following lemma.

Lemma 2.3.5. *Let* $\mathcal{X}$ *be a Mal'cev basis. If the* $\mathbb{Q}$*-span* $\bigoplus_{j=1}^m \mathbb{Q}X_j$ *contains a lattice* $\exp^{-1}\Gamma$*, then it is also the* $\mathbb{Q}$*-span of* $\exp^{-1}\Gamma$*. In particular, an element* $g \in G$ *(resp.* $X \in \mathfrak{g}$*) is rational if and only if* $\exp^{-1} g$ *(resp.* $X \in \mathfrak{g}$*) is in the* $\mathbb{Q}$*-span of* $\exp^{-1}\Gamma$*.*

Proof. Suppose otherwise, then $\exp^{-1}\Gamma$ is contained in a proper $\mathbb{Q}$ vector subspace in $\bigoplus_{j=1}^m \mathbb{Q}X_j$. The $\mathbb{R}$-span of this subspace is a proper subspace of $\mathfrak{g}$. This contradicts the Zariski density of Γ. □

Definition 2.3.6. *We say a Mal'cev basis* $\mathcal{X}$ *is* **rational** *if it is as in 2.3.5, or equivalently, if* $\mathcal{X} \subset \mathfrak{g}_{\mathbb{Q}}$*.*

Proposition 2.3.7. If Γ is a lattice in a simply connected nilpotent Lie group G, then there are two lattices Λ_+, Λ_- in the vector space $\mathfrak{g}$, such that $\Lambda_- \subseteq \exp^{-1}\Gamma \subseteq \Lambda_+$.

Here Λ_+, Λ_- are lattices in the sense of the usuall additive group structure.

Proof. The existence of Λ_+ is contained by Claim 2.3.2.(3). We will construct Λ_-.

We keep the notations from the proof of the "only if" direction of Theorem 2.3.1 and follow the steps there to define the rational Mal'cev basis $\mathcal{X}$ adapted to the lower central series filtration $\{\mathfrak{g}_{(i)}\}_{i=1}^s$. By the proof of the "if" direction of that same theorem, there is a positive integer R such that $\exp(\bigoplus_{j=1}^m R\mathbb{Z}X_j)$ is a lattice $\widetilde{\Gamma}$ of G. The collection $\widetilde{\mathcal{X}} = \{RX_1, \cdots, RX_m\}$ is also a Mal'cev basis adapted to $\{\mathfrak{g}_{(i)}\}_{i=1}^s$. Moreover, by diagram (1.14), $\exp^{-1}(\widetilde{\Gamma}_{(i)}/\widetilde{\Gamma}_{(j)})$ is the lattice generated by $RX_{m-m_i+1}, \cdots, RX_{m-m_{i+1}}$ in $\mathfrak{g}_{(i)}/\mathfrak{g}_{(i+1)}$. Then Claim 2.3.2 shows that Γ and $\widetilde{\Gamma}$ are respectively generated by $\{\exp X_j, : 1 \le j \le m\}$ and $\{\exp(RX_j), : 1 \le j \le m\}$. In particular, $\widetilde{\Gamma} \subseteq \Gamma$ because $\exp(RX_j) = (\exp X_j)^R$. Therefore for $\Lambda_- = \bigoplus_{j=1}^m R\mathbb{Z}X_j$, $\Lambda_- = \exp^{-1}\widetilde{\Gamma} \subseteq \exp^{-1}\Gamma$. □

This allows to give a more intrinsic definition of rational elements of G.

Corollary 2.3.8. *An element $g \in G$ is a rational element if and only if there exists $n \in \mathbb{N}$ such that $g^n \in \Gamma$.*

Proof. Suppose $g^n = \gamma \in \Gamma$. Then $\exp^{-1} g = \frac{1}{n} \exp^{-1} \gamma$ is a rational vector, and thus g is a rational element.

Conversely, suppose g is rational, then $g = \exp X$ for a rational vector $X \in \mathfrak{g}_{\mathbb{Q}}$. Let Λ_- be as in Proposition 2.3.7. As Λ_- is a rational lattice in $\mathfrak{g}_{\mathbb{Q}}$, there exists n such that $nX \in \Lambda_-$. Thus $g^n = \exp(nX) \in \exp \Lambda_- \subseteq \Gamma$. □

Definition 2.3.9. *Suppose G is a simply connected nilpotent Lie group admitting a lattice Γ. A connected closed subgroup H of G is rational if its Lie algebra $\mathfrak{h}$ is a rational subspace with respect to a rational Mal'cev basis $\mathcal{X}$.*

Lemma 2.3.5 shows that the rationality of H does not depending on the choice of $\mathcal{X}$.

Corollary 2.2.8 can be generalized to more subgroups.

Proposition 2.3.10. Suppose G, Γ are as above and H is a connected closed subgroup of G.

(1) H is rational if and only if $\Gamma \cap H$ is a lattice in H.

(2) If H is a connected rational closed normal subgroup of G, then $\Gamma/(\Gamma \cap H)$ is a lattice in G/H.

Proof. (1) Suppose H is a connected rational closed subgroup , then $\mathfrak{h}$ is a rational subspace in the coordinates given by $\mathcal{X}$. Hence the intersection $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$ is dense in $\mathfrak{h}$. Let Λ_- be as in Proposition 2.3.7. Then for every $X \in \mathfrak{h}_{\mathbb{Q}}$, there is $n \in \mathbb{N}$ such that $nX \in \Lambda_-$. In particular, $\mathfrak{h}$ has a linear basis consisting of elements of $\Lambda_- \subseteq \exp^{-1} \Gamma$. So $\exp^{-1}(\Gamma \cap H)$ is not contained in any proper subspace of $\mathfrak{h}$. By Theorem 2.2.6, this implies that $\Gamma \cap H$ is a lattice in H.

Conversely, if $\Gamma \cap H$ is a lattice in H. Then $\mathfrak{h}$ is linearly spanned by $\exp^{-1}(\Gamma \cap H) \subseteq \exp^{-1} \Gamma \subset \mathfrak{g}_{\mathbb{Q}}$. Thus $\mathfrak{h}$ is a rational subspace of $\mathfrak{g}$ and H is rational.

(2) For a connected rational closed subgroup H, knowing that $\Gamma \cap H$ is a lattice in H is sufficient for applying the same argument in the proof of implication (2)⇒(5) in Theorem 2.2.6 to show the second part of the statement. □

The proposition, together with Corollary 2.2.8 implies:

Corollary 2.3.11. *In Corollary 2.2.8, the subgroups $G_{(i)}$, $1 \leq i \leq s$ are rational.*

Exercises

Exercise 2.3.1. Does every simply connected nilpotent Lie group has a lattice?

Exercise 2.3.2. Show that: in a simply connected nilpotent Lie group G with a lattice Γ, if H is a rational connected closed subgroup of G then so is the normalizer $N_G(H)$.

2.4 Nilmanifolds

Definition 2.4.1. *A* **compact nilmanifold** *is the quotient G/Γ of a simply connected nilpotent Lie group by a lattice Γ.*

By Theorem 2.2.6, such a quotient is indeed compact. Moreover, because of the discreteness of Γ, G/Γ is a manifold.

In a more general definition, a nilmanifold space is a manifold that admits a transitive action by a simply connected nilpotent Lie group G. In other words, a nilmanifold is a homogeneous space of the form G/H where H is a closed subgroup. A theorem of Mal'cev [Mal51] asserts that if a nilmanifold is compact, then it has the form of Definition 2.4.1.

Definition 2.4.2. *A* **compact infranilmanifold** *is a manifold finitely covered by a compact nilmanifold.*

Example 2.4.3. The torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is a compact nilmanifold. The Klein bottle is a compact infranilmanifold as it is double covered by $\mathbb{T}^2$.

Suppose $M = G/\Gamma$ is a compact nilmanifold. From now one denote $\Gamma_{(i)} = \Gamma \cap G_{(i)}$. Then the quotients $G_{(i)}/\Gamma_{(i)}$ and $G/G_{(i)}\Gamma = (G/G_{(i)})\big/(\Gamma/\Gamma_{(i)})$ are also compact nilmanifolds. More generally, suppose $\{\mathfrak{g}_i\}_{i=1}^r$ is a filtration of $\mathfrak{g}$ such that each $\mathfrak{g}_i$ is a rational subspace of $\mathfrak{g}$. Denote $G_i = \exp \mathfrak{g}_i$ and $\Gamma_i = \Gamma \cap G_i$. Then by Proposition 2.3.10, G_i/Γ_i and $G/G_i\Gamma_i$ are compact nilmanifolds. Moreover, there is a chain of natural projections

$$G/\Gamma = G/G_{r+1}\Gamma \to G/G_r\Gamma \to \cdots \to G/G_1\Gamma = \{\mathrm{pt}\}. \tag{2.3}$$

Furthermore, as $\{\mathfrak{g}_j\}_{j \geq i}$ is a filtration of $\mathfrak{g}_i$, for every $j \geq i$, Γ_i/Γ_j is a lattice in G_i/G_j, and $G_i/G_j\Gamma_i = (G_i/G_j)\big/(\Gamma_i/\Gamma_j)$ is a compact nilmanifold.

Note that when $j = i + 1$, G_i/G_{i+1} is a simply connected abelian Lie group and thus isomorphic to $\mathbb{R}^{d_i}$. Two points x, x' in $G/G_{i+1}\Gamma = (G/G_{i+1})\big/(\Gamma/\Gamma_{i+1})$ have the same projection in $G/G_i\Gamma$ if and if there is an element $h \in G_i/G_{i+1}$ such that $x' = hx$. In other words, the preimage in $G/G_{i+1}\Gamma$ of every $y \in G/G_i\Gamma$ is an orbit of the abelian subgroup G_i/G_{i+1} of the simply connect nilpotent Lie group G/G_{i+1}.

Furthermore, for all x and h as above, $hx = x$ if and only if there is $f \in G/G_{i+1}$ and $\eta \in \Gamma/\Gamma_{i+1}$ such that $x = f\Gamma_{i+1}$ and $hf = f\eta$. Because $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$, $h \in G_i/G_{i+1}$ is in the center of G/G_{i+1}. Therefore $fh = hf = f\eta$, or equivalently $h = \eta \in \Gamma/\Gamma_{i+1}$. Therefore $h \in G_i/G_{i+1} \cap \Gamma/\Gamma_{i+1} = \Gamma_i/\Gamma_{i+1}$. In otherwords, the stablizer of $x \in G/G_{i+1}\Gamma$ is (Γ_i/Γ_{i+1}), which implies the preimage $\pi^{-1}(y)$ in $G/G_{i+1}\Gamma$ of every $y \in G/G_i\Gamma$ is homeomorphic to the compact quotient $(G_i/G_{i+1})\big/(\Gamma_i/\Gamma_{i+1})$. Note that this quotient is actually a compact abelian group isomorphic to $\mathbb{T}^{d_i} = \mathbb{R}^{d_i}/\mathbb{Z}^{d_i}$. The action by G_i/G_{i+1} on $G/G_{i+1}\Gamma$ degenerates into a free action of $(G_i/G_{i+1})\big/(\Gamma_i/\Gamma_{i+1})$, and every orbit is the preimage of a point in $G/G_i\Gamma$.

In particular, the quotient $G/G_2\Gamma$ and $G_{(s)}/\Gamma_{(s)}$ are both tori, respectively isomorphic to $\mathbb{T}^{d_1}$ and $\mathbb{T}^{d_s}$. They are respectively a factor and a subset of G/Γ, and are repsectively called the **horizontal torus** and **vertical torus** of G/Γ. They will play important roles in the study of dynamics on G/Γ.

Definition 2.4.4. *A* **fiber bundle** *is a tuple (X, B, F, π), where X, B, F are a topological space and $\pi : X \to B$ is a continuous surjective map, such that there is an open covering $\{U\}_{U\in\mathcal{U}}$ of B, and for every U, there is a homeomorphism $\phi_U : U \times F \to \pi^{-1}(U)$ such that the following diagram commute:*

$$\begin{array}{ccc} U \times F & \xrightarrow{\phi_U} & \pi^{-1}(U) \\ \Big\downarrow{\scriptstyle \pi_U} & & \Big\downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\text{id}} & U \end{array} \tag{2.4}$$

where π_U is the projection from $U \times F$ to U.

The spaces B and F are respectively called the **base** and **fiber** of X.

Definition 2.4.5. *For a topological group G, a left* **principal G-bundle** *is a fiber bundle (X, B, π) equipped with a continuous free left G-action of X that preserves the fibers and acts transitively on each of them.*

In this case, the fiber of X is homeomorphic to G.

The discussion we had earlier in fact says:

Proposition 2.4.6. Suppose Γ is a lattice in a simply connected nilpotent Lie group G, and $\{\mathfrak{g}_i\}_{i=1}^r$ is a filtration of $\mathfrak{g}$ consisting of rational subspaces of $\mathfrak{g}$ (with respect to the $\mathbb{Q}$-structure determined by Γ). Then, with notations above, G has the structure of a tower of principal torus bundles given by (2.3), where $G/G_{i+1}\Gamma$ is a principal $(G_i/G_{i+1})\big/(\Gamma_i/\Gamma_{i+1})$-bundle over $G/G_i\Gamma$.

Example 2.4.7. Suppose $G = H_{2n+1}$ and Γ are respectively the Heisenberg Lie group and the lattice Γ from Example 2.2.7. Then $G_{(2)} = \{(0,0,z) : z \in \mathbb{R}\} \cong \mathbb{R}$ and $\Gamma_{(2)} = \{(0,0,z) : z \in \mathbb{Z}\} \cong \mathbb{Z}$. The quotient $G/G_{(2)}$ is parametrized by $\{(\mathbf{x},\mathbf{y}) : \mathbf{x},\mathbf{y} \in \mathbb{R}^n\} \cong \mathbb{R}^{2n}$, and its lattice $\Gamma/\Gamma_{(2)}$ is identified with $\{(\mathbf{x},\mathbf{y}) : \mathbf{x},\mathbf{y} \in \mathbb{R}^n\} \cong \mathbb{T}^{2n}$. Thus the projection $G/\Gamma \to G/G_{(2)}\Gamma$ gives the compact nilmanifold G/Γ the structure of a principal $\mathbb{T}^1$-bundle over $\mathbb{T}^{2n}$.

Exercises

Exercise 2.4.1. Show that the $2n+1$ dimensional Heisenberg nilmanifold from Example 2.4.7 is also a $\mathbb{T}^{n+1}$-fiber bundle over $\mathbb{T}^n$, but this bundle structure is not a principal one.

Exercise 2.4.2. Show that a G-bundle is a left principal bundle if and only if there exist open charts $\{U\}_{U\in\mathcal{U}}$ and homeomorphisms ϕ_U as in (2.4), such that for each non-empty intersection $U \cap V$, $U, V \in \mathcal{U}$, the transition map $\phi_U^{-1} \circ \phi_V : (U\cap V) \times G \to (U \cap V) \times G$ has the form $(x, g) \to (x, h(x)g)$ where h is a continuous map from $U \cap V$ to G.

Exercise 2.4.3. Describe the transition maps in Exercise 2.4.1.