Name: ______________________________ ID: ______________

1. Find the maximum value of

$$P = 2pq + 2pr + 2rq$$

subject to the constraint $p + q + r = 1$.

$P$ represents the probability of that you get two different colors when you draw from a jar with purple, quartz, and red marbles, where the proportion of each color of marble is $p$, $q$, and $r$, respectively. For a more relevant example, replace "marbles" with "genes that determine blood type" and "red, green, and blue" with "A, B, and O."

**Solution:** Using our critical point techniques, we set $r = 1 - p - q$, and convert $P$ to a function of two variables, so

$$P = 2pq + 2p(1 - p - q) + 2(1 - p - q)q.$$

Then, using the product rule

$$\frac{\partial P}{\partial p} = 2q + 2(1 - p - q) - 2p - 2q = 2(1 - 2p - q),$$
$$\frac{\partial P}{\partial q} = 2p - 2p - 2q + 2(1 - p - q) = 2(1 - p - 2q).$$

Setting both of these to zero, we solve for the critical points

$$2p + q = 1,$$
$$p + 2q = 1,$$

from which we conclude that $p = \frac{1}{3}$ and $q = \frac{1}{3}$, as we might expect from symmetry. We then evaluate $P$. It is simpler to compute first that $r = 1 - p - q = \frac{1}{3}$, which we plug into the original formula to find that the maximum value of $P$ is

$$P = \frac{2}{9} + \frac{2}{9} + \frac{2}{9} = \frac{2}{3}.$$

Alternatively, using Lagrange multipliers, we go back to the original formula of $P$ as a function of three variables, and compute

$$\nabla P = \langle 2q + 2r, 2p + 2r, 2p + 2q\rangle.$$

Setting $g(p, q, r) = p + q + r$ for the constraint, we see that

$$\nabla g = \langle 1, 1, 1\rangle.$$

To find the extreme values, we set $\nabla P = \lambda \nabla g$. Along with the constraint, we solve the system

$$2q + 2r = \lambda,$$
$$2p + 2r = \lambda,$$
$$2p + 2q = \lambda,$$
$$p + q + r = 1.$$

Adding the first three equations together, we find that

$$4p + 4q + 4r = 3\lambda.$$

On the other hand, multiplying the last equation by four, we find that

$$4p + 4q + 4r = 4.$$

We conclude that $\lambda = \frac{4}{3}$. Plugging that in and dividing by two, we are left with the system

$$q + r = \frac{2}{3},$$
$$p + r = \frac{2}{3},$$
$$p + q = \frac{2}{3},$$

By adding any two of the equations together and subtracting the third, we find that $2p = \frac{2}{3}$, $2q = \frac{2}{3}$, and $2r = \frac{2}{3}$. We obtain the solution $p = q = r = \frac{1}{3}$, from which we conclude that the maximum value of $P$ is $\frac{2}{3}$, as before.

2. Let $R$ be the rectangle $[0,3] \times [1,2]$. Evaluate the integral

$$\iint_R x^2 y \, dA.$$

You may want to evaluate the integral in more than one way in order to check your work, but you'll get full credit if you just do it one way.

**Solution:** Integrating first with respect to $x$, we compute

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \tfrac{1}{3} x^3 y \Big|_{x=0}^3 \, dy = \int_1^2 9y \, dy = \tfrac{9}{2} y^2 \Big|_{y=1}^2 = \tfrac{9}{2}(4-1) = \tfrac{27}{2}.$$

Alternatively, integrating first with respect to $y$, we compute

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \tfrac{1}{2} x^2 y^2 \Big|_{y=1}^2 \, dx = \int_0^3 \tfrac{1}{2} x^2 (4-1) \, dx = \int_0^3 \tfrac{3}{2} x^2 \, dx = \tfrac{1}{2} x^3 \Big|_{x=0}^3 = \tfrac{27}{2}.$$

Alternatively, because $x^2 y = (x^2)(y)$ can be expressed as the product of a function that depends only on $x$ and a function that depends only on $y$, we can evaluate the integral via

$$\left( \int_0^3 x^2 \, dx \right) \left( \int_1^2 y \, dy \right) = \left( \tfrac{1}{3} x^3 \Big|_{x=0}^3 \right) \left( \tfrac{1}{2} y^2 \Big|_{y=1}^2 \right) = \tfrac{1}{3}(27-0)\tfrac{1}{2}(4-1) = \tfrac{27}{2}.$$