Math 253A Midterm 2

April 2, 2020

- Write your solutions on separate sheets of paper and upload them on Gradescope, just like your homework assignments.
- Only use the resources allowed on the exam honor code certification form.
- Be sure to include the exam honor code certification form with your solutions. If you are unable to print it, copy the form by hand.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.

1. Determine the set of points where the function

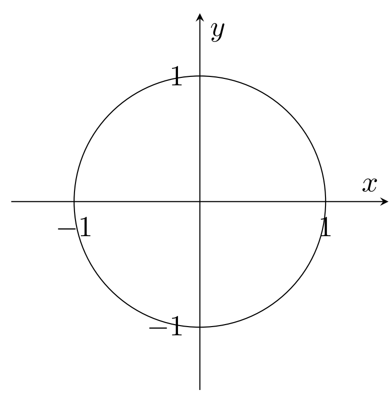
$$f(x,y) = \frac{1}{1 - x^2 - y^2}$$

is discontinuous. Draw and/or describe this set. Be specific.

Solution: The quotient of continuous functions is continuous except when the denominator is zero, so f is continuous except when $1 - x^2 - y^2 = 0$, or, equivalently, $x^2 + y^2 = 1$. The set of discontinuities is

$$\{(x,y) \mid x^2 + y^2 = 1\}.$$

This set is the circle of radius 1 centered at the origin.



2. Find the absolute maximum and minimum values of $f(x,y) = xy^2$ on the domain

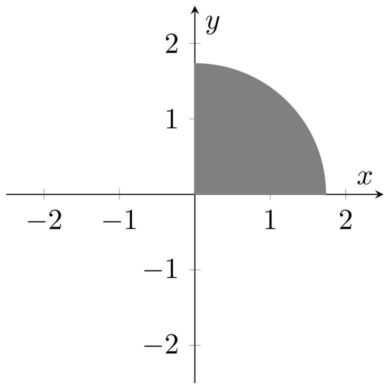
$$D = \left\{(x,y) \mid x \geq 0,\, y \geq 0,\, x^2 + y^2 \leq 3\right\}.$$

Solution: The maximum and minimum values occur either at critical points or on the boundary. We begin by finding critical points. We compute

$$\nabla f = \langle y^2, 2xy \rangle.$$

At a critical point, $\nabla f = \mathbf{0}$. Since $y^2 = 0$, we know $y = 0$. Then, $2xy = 0$ is automatically satisfied. Thus, any point on the x-axis is a critical point.

Before we continue, we sketch the region, using the fact that the circle has radius $\sqrt{3}$.



We note that our critical points, namely the x-axis, are all on the boundary. Thus, there are no critical points in the interior, so the absolute maximum and minimum must occur on the boundary.

On the x-axis, $y = 0$, so $f(x, y) = xy^2 = 0$. Likewise, on the y-axis, $x = 0$, so $f(x, y) = xy^2 = 0$.

The remaining part of the boundary is the arc of the circle $x^2+y^2 = 3$ in the first quadrant. One way to find the maximum value of f on the circle is by parametrizing the circle by

$$x = \sqrt{3}\cos\theta, \qquad y = \sqrt{3}\sin\theta, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

Then

$$f(x, y) = xy^2 = 3\sqrt{3}\cos\theta\sin^2\theta.$$

Taking the derivative of this expression with respect to θ using the product rule, we see that we have a critical point when

$$\begin{aligned}
0 &= 3\sqrt{3}\left(-\sin^3\theta + 2\cos^2\theta\sin\theta\right), \\
0 &= 3\sqrt{3}\sin\theta(2\cos^2\theta - \sin^2\theta), \\
0 &= 3\sqrt{3}\sin\theta(3\cos^2\theta - 1).
\end{aligned}$$

Thus, either $\sin\theta = 0$ or $\cos^2\theta = \frac{1}{3}$. In the first case, $\theta = 0$, so we are on the x-axis, which we have already considered. In the second case, $\sin^2\theta = 1 - \cos^2\theta = \frac{2}{3}$. Since we are in the first quadrant, we know that $\cos\theta = \frac{1}{\sqrt{3}}$ rather than $-\frac{1}{\sqrt{3}}$. We conclude that

$$f(x, y) = 3\sqrt{3}\cos\theta\sin^2\theta = 3\sqrt{3}\frac{1}{\sqrt{3}}\frac{2}{3} = 2.$$

Thus the maximum value of f on the domain D is 2, and the minimum value on this domain is 0.

Alternatively, we can find the maximum value using Lagrange multipliers. We set $g(x, y) = x^2 + y^2$, so $\nabla g = \langle 2x, 2y\rangle$. Then, we set $\nabla f = \lambda\nabla g$, so our system of equations is

$$\begin{aligned}
y^2 &= \lambda 2x, \\
2xy &= \lambda 2y, \\
x^2 + y^2 &= 3.
\end{aligned}$$

We'd like to divide the second equation by y, but first we must consider $y = 0$. In that case, the third equation gives $x^2 = 3$. Since $x \geq 0$, we know that $x = \sqrt{3}$. Then, the first equation $y^2 = \lambda 2x$ gives $0 = \lambda 2\sqrt{3}$, so $\lambda = 0$. All three equations are satisfied. But, as before, the point $(\sqrt{3}, 0)$ is on the x-axis, so we already know that $f = 0$ there.

If $y \neq 0$, then we can divide the second equation by $2y$, giving $\lambda = x$. Thus, $y^2 = 2x^2$. Substituting into the third equation, we find that $3x^2 = 3$, so $x = 1$, again using $x \geq 0$ to rule out $x = -1$. We conclude from either the first or third equation that $y^2 = 2$, and we compute that $f(x, y) = xy^2 = 2$, as before.

3. Let

$$f(x, y) = \ln(3x - y).$$

Find the second-degree Taylor polynomial of f at $(1, 2)$.

Solution: We need to compute the values of f and its partial derivatives at $(1, 2)$. We use the chain rule.

$$\begin{aligned} f(1,2) &= \ln(3-2) = \ln 1 = 0, \\ f_x(1,2) &= \frac{3}{3x-y} = 3, \\ f_y(1,2) &= -\frac{1}{3x-y} = -1, \\ f_{xx}(1,2) &= -\frac{9}{(3x-y)^2} = -9, \\ f_{xy}(1,2) &= \frac{3}{(3x-y)^2} = 3, \\ f_{yy}(1,2) &= -\frac{1}{(3x-y)^2} = -1. \end{aligned}$$

Thus, using the formula, we find that the second-degree Taylor polynomial of f at $(1, 2)$ is

$$Q(x, y) = 0 + 3(x-1) - (y-2) - \tfrac{9}{2}(x-1)^2 + 3(x-1)(y-2) - \tfrac{1}{2}(y-2)^2.$$

4. (a) Compute

$$\int_0^2 y\sqrt{x+2}\, dx.$$

Solution: Treating y as a constant, we compute

$$\int_0^2 y\sqrt{x+2}\, dx = \tfrac{2}{3}y(x+2)^{3/2}\Big|_{x=0}^{2} = \tfrac{2}{3}y\left(4^{3/2} - 2^{3/2}\right) = \tfrac{2}{3}y\left(8 - 2\sqrt{2}\right).$$

(b) Compute

$$\int_0^3 y\sqrt{x+2}\,dy.$$

Solution: Treating x as a constant, we compute

$$\int_0^3 y\sqrt{x+2}\,dy = \tfrac{1}{2}y^2\sqrt{x+2}\Big|_{y=0}^{3} = \tfrac{9}{2}\sqrt{x+2}.$$