Math 253A Midterm 1

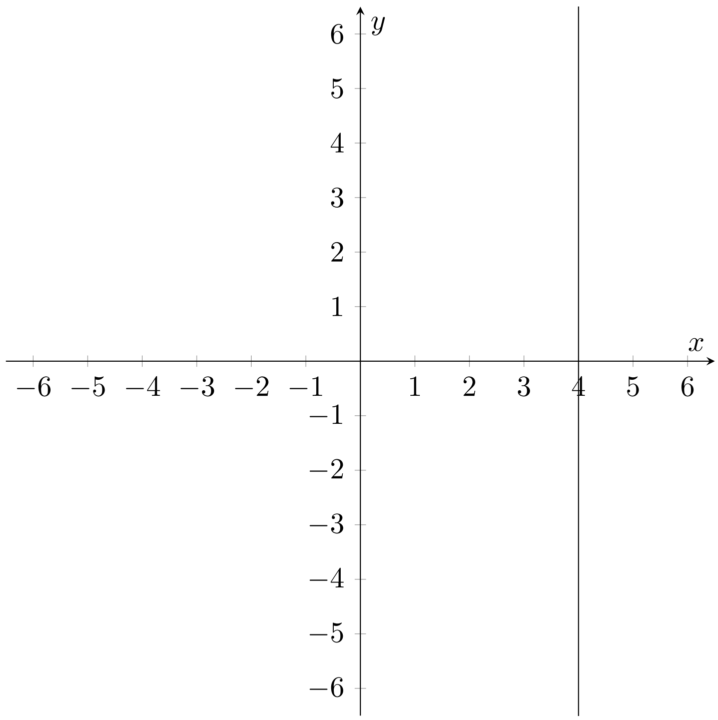
February 13, 2020

Name: ______________________________ ID: ____________________

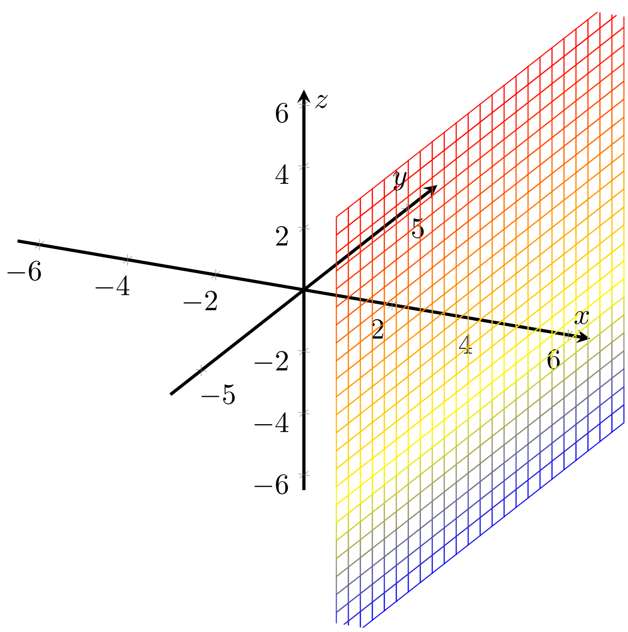
- Each page has a space at the top for the last 4 digits of your student ID. Make sure that you fill that out on at least one side of every sheet of paper.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.
- You may not use any tools or resources other than writing implements. In particular, no calculators, phones, notes, and so forth.

1. What does the equation $x = 4$ represent in $\mathbb{R}^2$? What does it represent in $\mathbb{R}^3$? Illustrate with sketches.

Solution: In two dimensions, the equation $x = 4$ represents a vertical line.

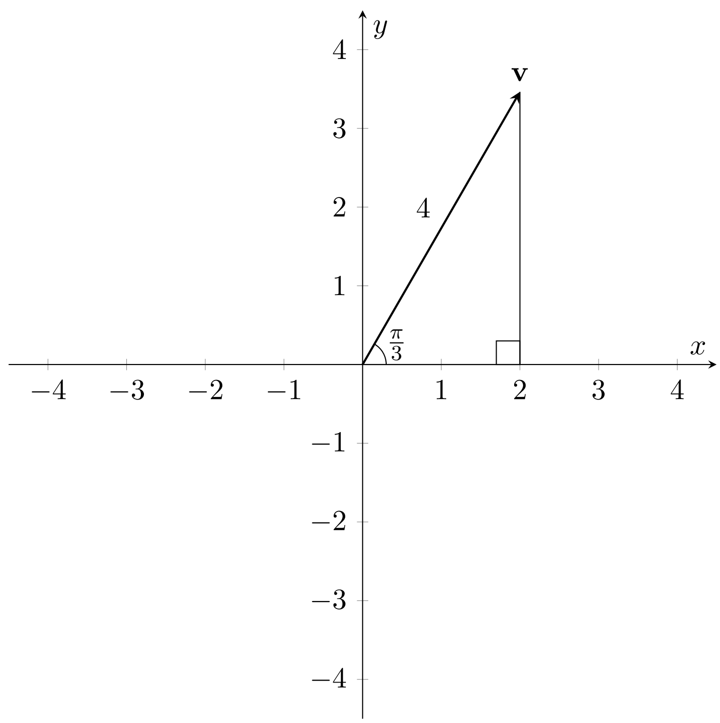


In three dimensions, the equation $x = 4$ represents a vertical plane parallel to the yz-plane.



2. Let $\mathbf{v}$ be the vector in the first quadrant that makes an angle of $\pi/3$ with the positive x-axis and has magnitude $|\mathbf{v}| = 4$. Write down $\mathbf{v}$ in component form.

Solution: A quick sketch can help us with this problem



Using trigonometry or geometry, we see that the horizontal segment has length 2, and the vertical segment has length $2\sqrt{3}$, so, in component form, we can write $\mathbf{v} = \langle 2, 2\sqrt{3} \rangle$ or $\mathbf{v} = 2\mathbf{i} + 2\sqrt{3}\mathbf{j}$.

3. Let $\mathbf{a} = \langle -1, 4, 8 \rangle$ and let $\mathbf{b} = \langle 12, 1, 2 \rangle$.

   (a) Find the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$.

   Solution: We compute the scalar projection

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= -12 + 4 + 16 = 8, \\ |\mathbf{a}| &= \sqrt{1 + 16 + 64} = 9, \\ \operatorname{comp}_{\mathbf{a}} \mathbf{b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{8}{9}. \end{aligned}$$

   (b) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

   Solution: We compute the vector projection by computing the vector in direction $\mathbf{a}$ with size $\frac{8}{9}$, which is

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{8}{9} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{8}{81} \langle -1, 4, 8 \rangle.$$

4. Consider the four points $P(-2,1,0)$, $Q(2,3,2)$, $R(1,4,-1)$, and $S(3,6,1)$. Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS.

Solution: We compute the three displacement vectors

$$\overrightarrow{PQ} = \langle 4,2,2\rangle,$$
$$\overrightarrow{PR} = \langle 3,3,-1\rangle,$$
$$\overrightarrow{PS} = \langle 5,5,1\rangle.$$

Then, we know that the volume of the parallelepiped spanned by these three vectors is the triple product $\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS})$, which we can compute using the determinant

$$\begin{aligned}\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS}) &= \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} \\ &= (4)(3)(1)+(2)(-1)(5)+(2)(3)(5)-(4)(-1)(5)-(2)(3)(1)-(2)(3)(5) \\ &= 12-10+30+20-6-30 = 16.\end{aligned}$$

Thus, the volume of the parallelepiped is 16.

5. Find the equation of the plane that contains the line

$$x = 1 + t, \qquad y = 2 - t, \qquad z = 4 - 3t$$

and is parallel to the plane

$$5x + 2y + z = 1.$$

Solution: The normal vector of the plane $5x + 2y + z = 1$ is $\mathbf{n} = \langle 5, 2, 1 \rangle$. Because our plane is parallel to this plane, $\mathbf{n}$ is also a normal vector of our plane.

To find the equation of our plane, we also need a point on the plane. Plugging in a value of t such as $t = 0$ into the equation of the line, we find that $(1, 2, 4)$ is on the plane. Thus, our plane has equation

$$5(x - 1) + 2(y - 2) + (z - 4) = 0.$$

Optionally, we could simplify this equation to

$$5x + 2y + z = 13.$$

One way to check our work is to verify that all of the points on the line are in our plane, not just the $t = 0$ point. We compute that, for a point on the line,

$$\begin{aligned} &5(x - 1) + 2(y - 2) + (z - 4) \\ &\qquad = 5(1 + t - 1) + 2(2 - t - 2) + (4 - 3t - 4) = 5t - 2t - 3t = 0, \end{aligned}$$

as desired.

6. Consider the surface defined by the equation

$$-4x^2 - y^2 + z^2 = 1.$$

(a) Identify the horizontal traces of the surface in the planes $z = k$. If the answer depends on the value of k, be sure to specify which values of k give which answer.

Solution: Plugging in $z = k$, we find that

$$\begin{aligned} -4x^2 - y^2 + k^2 &= 1, \\ 4x^2 + y^2 &= k^2 - 1. \end{aligned}$$

This equation tells us that the trace is an ellipse if $k^2 - 1 > 0$, which happens when $k > 1$ or $k < -1$. If $k = 1$ or $k = -1$, then it is the equation $4x^2 + y^2 = 0$, which tells us that the trace is the single point $(0, 0)$. If $-1 < k < 1$, then the left-hand side of the equation is nonnegative, whereas the right-hand side of the equation is negative, so the equation has no solutions and the trace is empty.

(b) Identify the vertical traces of the surface in the planes $x = k$. If the answer depends on the value of k, be sure to specify which values of k give which answer.

Solution: Plugging in $x = k$, we find that

$$\begin{aligned} -4k^2 - y^2 + z^2 &= 1, \\ z^2 - y^2 &= 1 + 4k^2. \end{aligned}$$

Because $1 + 4k^2$ is never zero, this is the equation of a hyperbola for all values of k.

(c) Identify the vertical traces of the surface in the planes $y = k$. If the answer depends on the value of k, be sure to specify which values of k give which answer.

Solution: Plugging in $y = k$, we find that

$$\begin{aligned} -4x^2 - k^2 + z^2 &= 1, \\ z^2 - 4x^2 &= 1 + k^2. \end{aligned}$$

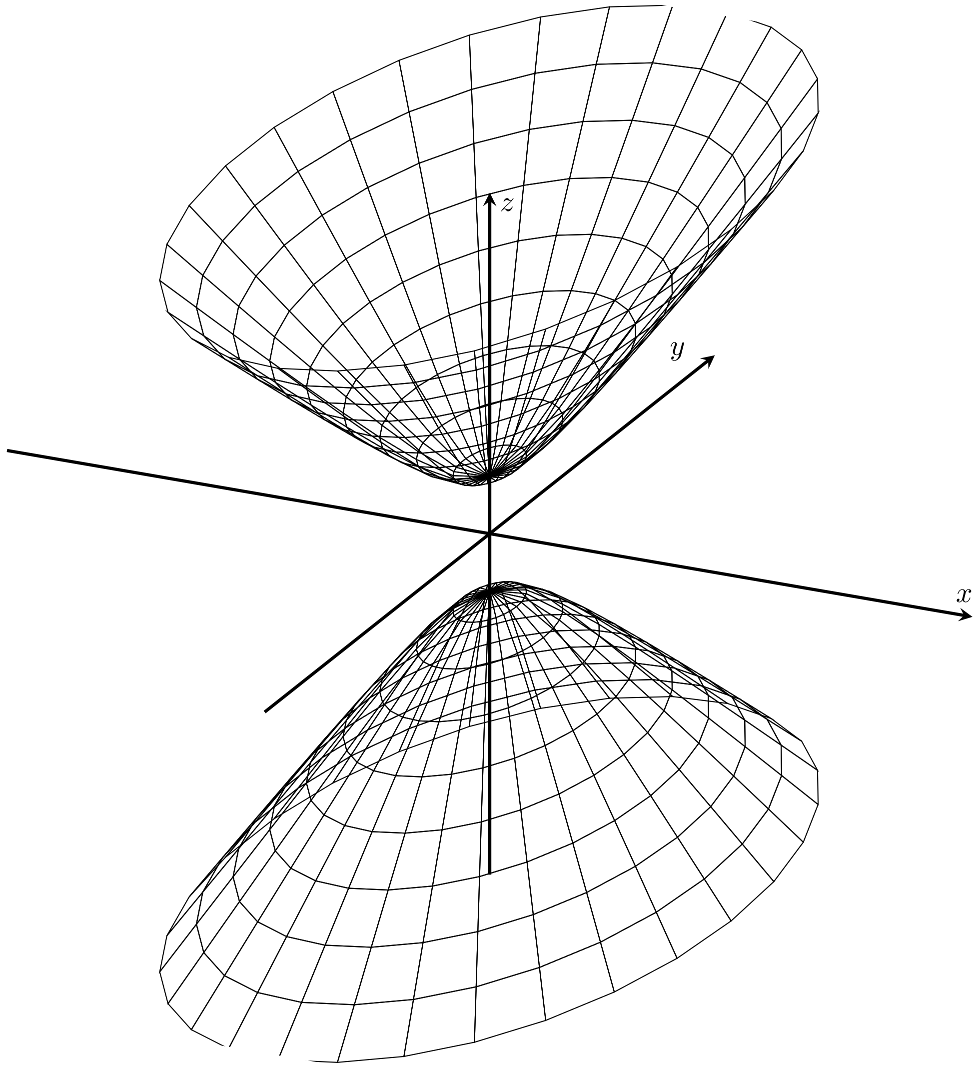
Because $1 + 4k^2$ is never zero, this is the equation of a hyperbola for all values of k.

(d) Identify the surface.

Solution: A surface with hyperbolic and elliptic traces is a hyperboloid or a cone. Because there is a gap when $-1 < z < 1$ that does not contain any part of the surface, this surface is a hyperboloid of two sheets.

(e) Sketch the surface. Your sketch does not need to be quantitatively correct, but it should show the correct type of surface in the correct location with the correct orientation. If you feel like you need to, feel free to write a sentence to clarify the location and orientation.

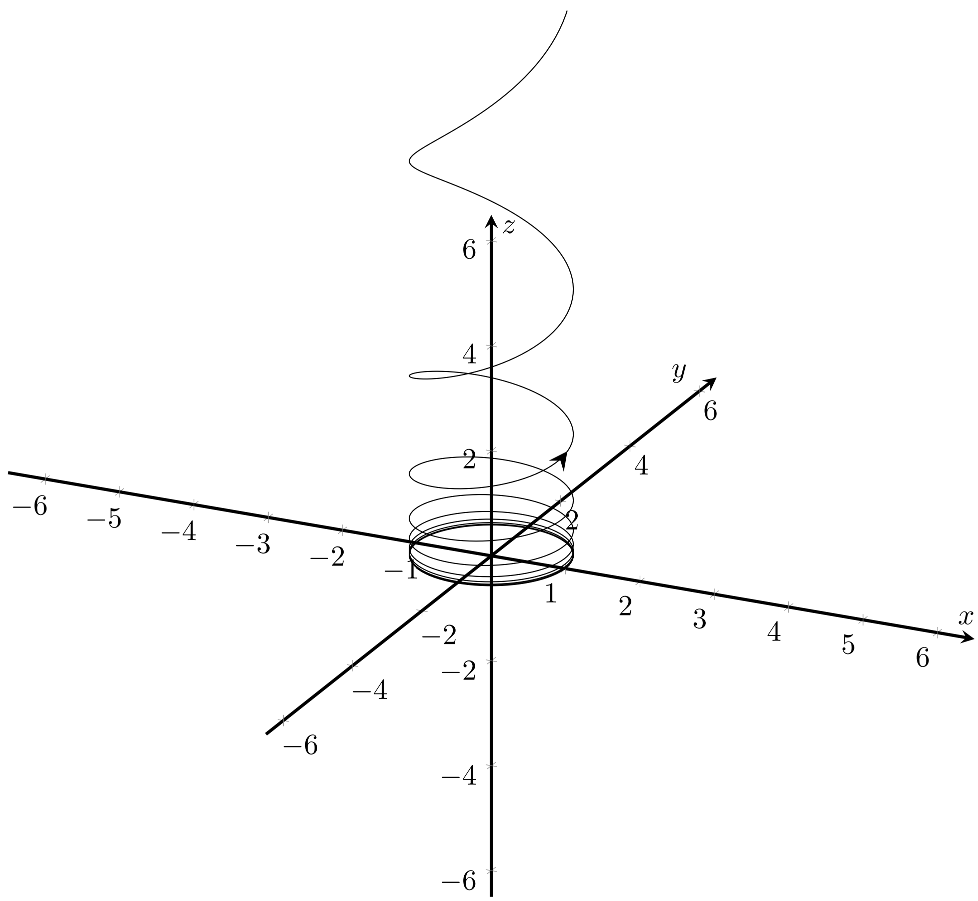
Solution: Based on the traces with $z = k$, we see that the hyperboloid opens in the positive and negative z directions, with a gap between the sheets when $-1 < z < 1$.



7. Consider the space curve defined by the equations

$$x = \cos 8t, \qquad y = \sin 8t, \qquad z = e^t.$$

Describe and sketch this curve. Make sure to specify the location, orientation, size, direction of travel as t increases, and any other salient features.



Solution: We see that the x and y coordinates parametrize a circle of radius 1, because $x^2 + y^2 = \cos^2 8t + \sin^2 8t = 1$. Thus, the curve is some sort of counterclockwise spiral shape on the cylinder of radius 1 about the z-axis. What about the z coordinate? We see that $z = e^t > 0$ so the curve is entirely above the xy-plane. As t increases, z increases, so our direction of travel is upwards. Moreover, $z = e^t$ increases very slowly when t is very negative and very quickly when t is very positive, so we expect the curve to wind around lots of times without moving up very much when it is close to the xy-plane and to move upwards in fewer winds when it is further away from the xy-plane.

8. If $\mathbf{r}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, show that $\mathbf{r}(t) \times \mathbf{r}'(t) = \omega\mathbf{a} \times \mathbf{b}$.

Solution: We compute

$$\mathbf{r}'(t) = -\omega\mathbf{a}\sin\omega t + \omega\mathbf{b}\cos\omega t.$$

Using the distributive property for the cross product, we compute that

$$\begin{aligned}
\mathbf{r}(t) \times \mathbf{r}'(t) &= -\mathbf{a}\cos\omega t \times \omega\mathbf{a}\sin\omega t + \mathbf{a}\cos\omega t \times \omega\mathbf{b}\cos\omega t \\
&\quad -\mathbf{b}\sin\omega t \times \omega\mathbf{a}\sin\omega t + \mathbf{b}\sin\omega t \times \omega\mathbf{b}\cos\omega t \\
&= -\omega\cos\omega t\sin\omega t\,\mathbf{a} \times \mathbf{a} + \omega\cos^2\omega t\,\mathbf{a} \times \mathbf{b} \\
&\quad -\omega\sin^2\omega t\,\mathbf{b} \times \mathbf{a} + \omega\sin\omega t\cos\omega t\,\mathbf{b} \times \mathbf{b} \\
&= \omega\cos^2\omega t\,\mathbf{a} \times \mathbf{b} - \omega\sin^2\omega t\,\mathbf{b} \times \mathbf{a} \\
&= \omega\cos^2\omega t\,\mathbf{a} \times \mathbf{b} + \omega\sin^2\omega t\,\mathbf{a} \times \mathbf{b} \\
&= \omega(\cos^2\omega t + \sin^2\omega t)\,\mathbf{a} \times \mathbf{b} \\
&= \omega\mathbf{a} \times \mathbf{b}.
\end{aligned}$$

9. Consider the graph of the function $y = e^x$.

(a) Compute the curvature of this curve. The book contains the formula

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

which you are welcome to use or not use.

Solution: In vector form, we can describe this curve as

$$\mathbf{r}(t) = \langle t, e^t \rangle.$$

We can then compute

$$\begin{aligned} \mathbf{r}'(t) &= \langle 1, e^t \rangle, \\ \mathbf{r}''(t) &= \langle 0, e^t \rangle, \\ \mathbf{r}'(t) \times \mathbf{r}''(t) &= \big((1)(e^t) - (e^t)(0)\big)\,\mathbf{k} = e^t\,\mathbf{k}, \\ \big|\mathbf{r}'(t) \times \mathbf{r}''(t)\big| &= e^t, \\ \big|r'(t)\big| &= \sqrt{1 + e^{2t}}, \\ \kappa &= \frac{e^t}{(1+e^{2t})^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}. \end{aligned}$$

Alternatively, if we remember the formula $\kappa = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$, we could obtain the same answer more quickly.

(b) What happens to the curvature as $x \to -\infty$?

Solution: When $x \to -\infty$, $e^x \to 0$, so $\kappa \to \frac{0}{1^{3/2}} = 0$. This makes sense because the curve is close very close to the x-axis, a straight line wtih no curvature.

(c) What happens to the curvature as $x \to \infty$?

Solution: As $x \to \infty$, we have that

$$0 \le \frac{e^x}{(1+e^{2x})^{3/2}} \le \frac{e^x}{(e^{2x})^{3/2}} = \frac{e^x}{e^{3x}} = e^{-2x} \to 0.$$

Thus, $\kappa \to 0$ as $x \to \infty$.

(d) At what point does the curve have maximum curvature?

Solution: To find the maximum value of κ, we look for the critical points of $\kappa(x)$. We compute that

$$\begin{aligned} \kappa'(x) &= e^x\left(1+e^{2x}\right)^{-3/2} + e^x\left(-\tfrac{3}{2}\right)\left(1+e^{2x}\right)^{-5/2} 2e^{2x} \\ &= e^x\left(1+e^{2x}\right)^{-5/2}\left((1+e^{2x}) - 3e^{2x}\right) \\ &= e^x\left(1+e^{2x}\right)^{-5/2}\left(1-2e^{2x}\right) \end{aligned}$$

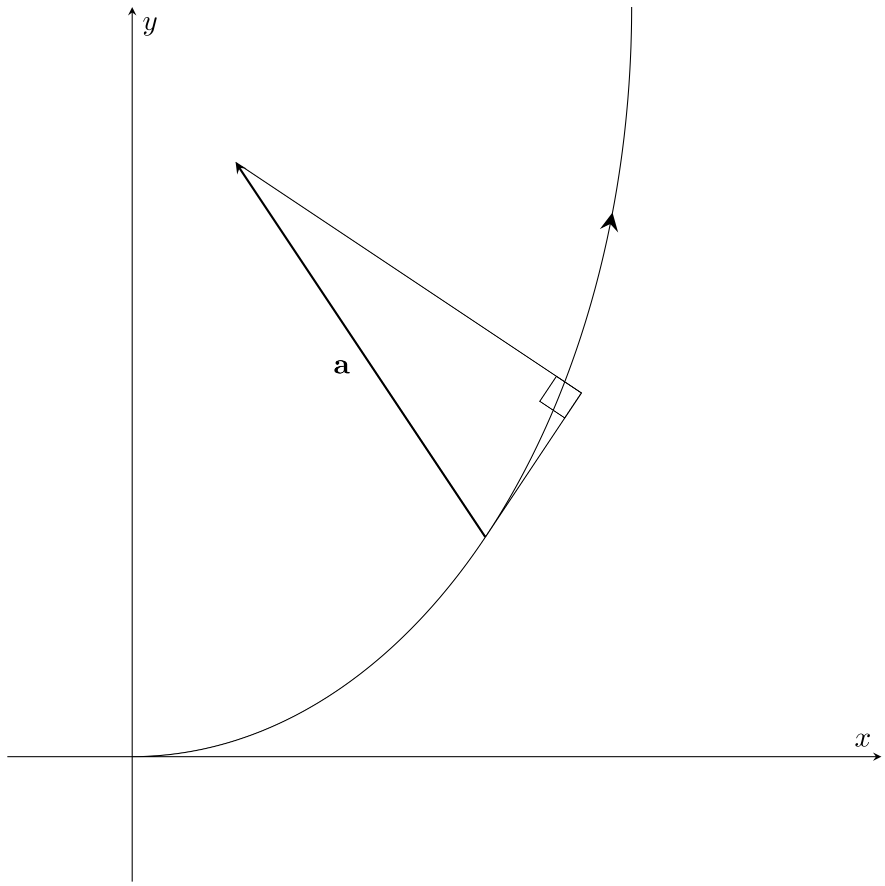
We have a critical point when $\kappa'(x) = 0$. The first two factors of our expression for $\kappa'(x)$ are positive, so our critical point occurs when

$$\begin{aligned} 1 &= 2e^{2x}, \\ e^{2x} &= \tfrac{1}{2}, \\ 2x &= \ln\tfrac{1}{2}, \\ x &= \tfrac{1}{2}\ln\tfrac{1}{2} = -\tfrac{1}{2}\ln 2, \\ y &= e^x = \sqrt{\tfrac{1}{2}} = \tfrac{\sqrt{2}}{2}. \end{aligned}$$

Thus, the critical point occurs at $(x, y) = \left(-\frac{1}{2}\ln 2, \frac{\sqrt{2}}{2}\right)$. Since this is the only critical point, we know that the maximum value of κ occurs here.
The problem does not ask for the maximum value of κ, but we can easily compute it by substituting $\frac{\sqrt{2}}{2}$ for e^x. We obtain that the maximum value of κ is

$$\kappa_{\max} = \frac{\frac{\sqrt{2}}{2}}{\left(1 + \frac{1}{2}\right)^{3/2}} = \frac{\frac{\sqrt{2}}{2}}{\frac{3\sqrt{3}}{2\sqrt{2}}} = \frac{2\sqrt{3}}{9}.$$

10. The magnitude of the acceleration vector **a** is $10\,\text{cm/s}^2$. Use the figure to estimate the tangential and normal components of **a**.



Solution: We construct the right triangle above to decompose the acceleration vector **a** into the component tangent to the curve and the component normal to the curve.

From the picture, it looks like the short leg of the triangle is somewhere between a third and a half of the length of **a**, so the tangential component a_T has size somewhere between $3\,\text{cm/s}^2$ and $5\,\text{cm/s}^2$. Because the tangential projection of **a** points in the same direction as the direction of travel along the curve, we know that $a_T > 0$. Thus, a reasonable estimate is $a_T \approx 4\,\text{cm/s}^2$.

To find the normal component, we use the Pythagorean theorem, finding that $a_N = \sqrt{10^2 - a_T^2}$. So, for example, if we estimated a_T to be $4\,\text{cm/s}^2$, then we would estimate that $a_N \approx \sqrt{10^2 - 4^2} = \sqrt{84} \approx \sqrt{81} = 9$, in cm/s^2.