Math 253A Final

May 14, 2020

- Write your solutions and upload them on Gradescope, just like your homework assignments. You can write your solutions on the exam pages or on separate sheets of paper, your choice.
- Only use the resources allowed on the exam honor code certification form.
- Be sure to include the exam honor code certification form with your solutions. If you are unable to print it, copy the form by hand.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.

1. Consider the part of the surface $z = 1 + x^2y^2$ that lies above the disk $x^2 + y^2 \le 4$.

   Set up a double integral in polar coordinates to compute its area. Do not compute antiderivatives, but do as much work as you can up to that point.

Solution: We compute

$$\nabla z = \langle 2xy^2, 2x^2y \rangle,$$
$$|\nabla z|^2 = 4x^2y^4 + 4x^4y^2 = 4x^2y^2(x^2 + y^2).$$

Thus, letting D be the disk of radius 2 about the origin, the area of the surface is

$$\begin{aligned} A &= \iint_D \sqrt{1 + |\nabla z|^2}\, dA \\ &= \iint_D \sqrt{1 + 4x^2y^2(x^2 + y^2)}\, dA \\ &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4(r\cos\theta)^2(r\sin\theta)^2r^2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^2 \sqrt{1 + r^6\sin^2 2\theta}\, r\, dr\, d\theta. \end{aligned}$$

2. Consider a brick with constant density ρ and side lengths a, b, and c. Place the brick so that its center of mass is at the origin, the sides of length a are parallel to the x-axis, the sides of length b are parallel to the y-axis, and the sides of length c are parallel to the z-axis.

   Find the moments of inertia of the brick about each of the coordinate axes.

Solution: Being careful with factors of 2, the brick occupies the space

$$E = \left\{ (x, y, z) \;\middle|\; -\frac{a}{2} \le x \le \frac{a}{2}, -\frac{b}{2} \le y \le \frac{b}{2}, -\frac{c}{2} \le z \le \frac{c}{2} \right\}.$$

Then, using formula 16 in 15.6, we have

$$\begin{aligned}
I_x &= \iiint_E (y^2 + z^2)\rho \, dV \\
&= \rho \left(\int_{-a/2}^{a/2} \int_{-b/2}^{b^2} \int_{-c/2}^{c/2} y^2 \, dz \, dx \, dy + \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_{-c/2}^{c/2} z^2 \, dz \, dx \, dy \right) \\
&= \rho \left(ac \int_{-b/2}^{b/2} y^2 \, dy + ab \int_{-c/2}^{c/2} z^2 \, dz \right) \\
&= \frac{\rho}{3} \left(ac \left(\left(\frac{b}{2}\right)^3 - \left(-\frac{b}{2}\right)^3 \right) + ab \left(\left(\frac{c}{2}\right)^3 - \left(-\frac{c}{2}\right)^3 \right) \right) \\
&= \frac{\rho}{12} \left(ab^3c + abc^3 \right) \\
&= \frac{\rho}{12} abc(b^2 + c^2).
\end{aligned}$$

The problem is the same if we swap x and y and we swap a and b. Likewise, the problem is the same if we swap x and z and we swap a and c. Thus, by symmetry,

$$\begin{aligned}
I_y &= \frac{\rho}{12} abc(a^2 + c^2) \\
I_z &= \frac{\rho}{12} abc(a^2 + b^2).
\end{aligned}$$

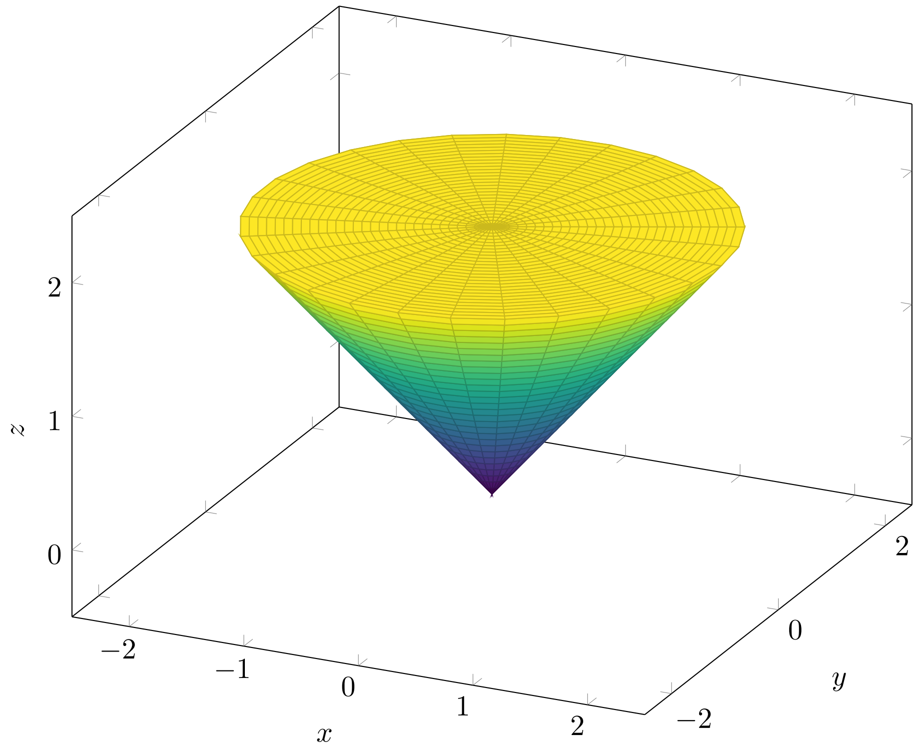
3. Consider the triple integral

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} z\,dz\,dx\,dy.$$

(a) Describe the domain of integration. Be specific. Use words or pictures to describe the shape qualitatively, and use numbers to describe the shape quantitatively.

Solution: The inequalities $-2 \leq y \leq 2$ and $-\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}$ are equivalent to the inequality $x^2+y^2 \leq 4$, or, equivalently, $r \leq 2$, so our solid lies above a disk of radius 2 centered at the origin. The inequality $\sqrt{x^2+y^2} \leq z \leq 2$ can be written as $r \leq z \leq 2$.

Drawing or imagining a picture, this region is a solid cone whose vertex is the origin and whose base is the disk defined by $r \leq 2, z = 2$.



(b) Express this integral in cylindrical coordinates.

Solution: Based on our work so far, the cylindrical integral can be written as

$$\int_{0}^{2\pi}\int_{0}^{2}\int_{r}^{2} z\,dz\,r dr\,d\theta.$$

(c) Evaluate the integral.

Solution: We compute

$$\begin{aligned}
\int_0^{2\pi}\int_0^2\int_r^2 z\,dz\,r\,dr\,d\theta &= 2\pi\int_0^2\left[\frac{1}{2}z^2\right]_{z=r}^{z=2} r\,dr \\
&= \pi\int_0^2 (4-r^2)r\,dr \\
&= \pi\int_0^2 (4r-r^3)\,dr \\
&= \pi\left[2r^2-\frac{1}{4}r^4\right]_0^2 \\
&= \pi(8-4) = 4\pi.
\end{aligned}$$

4. Evaluate

$$\iiint\limits_E y^2\,dV,$$

where E is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$.

Solution: In spherical coordinates, the first equation becomes $\rho \leq 3$, and the second equation becomes $\phi \leq \frac{\pi}{2}$.

Thus, rewriting the integral in spherical coordinates using equation 3 in Section 15.8, we compute

$$\begin{aligned}
\iiint\limits_E y^2\,dV &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 \rho^2 \sin^2\phi \sin^2\theta\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \left(\int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi\right)\left(\int_0^{2\pi} \sin^2\theta\, d\theta\right)\left(\int_0^3 \rho^4\, d\rho\right) \\
&= \left(\int_0^{\frac{\pi}{2}} (1-\cos^2\phi)\sin\phi\, d\phi\right)\left(\int_0^{2\pi} \frac{1}{2}\, d\theta\right)\left[\frac{\rho^5}{5}\right]_0^3 \\
&= \frac{243\pi}{5}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{2}} \\
&= \frac{243\pi}{5}\left((-0+0) - \left(-1+\frac{1}{3}\right)\right) \\
&= \frac{162\pi}{5}.
\end{aligned}$$

Here, we used the fact that the average value of $\sin^2\theta$ over a full period is $\frac{1}{2}$.

5. Compute the Jacobian of the polar coordinate transformation

$$x = u\cos v, \qquad y = u\sin v.$$

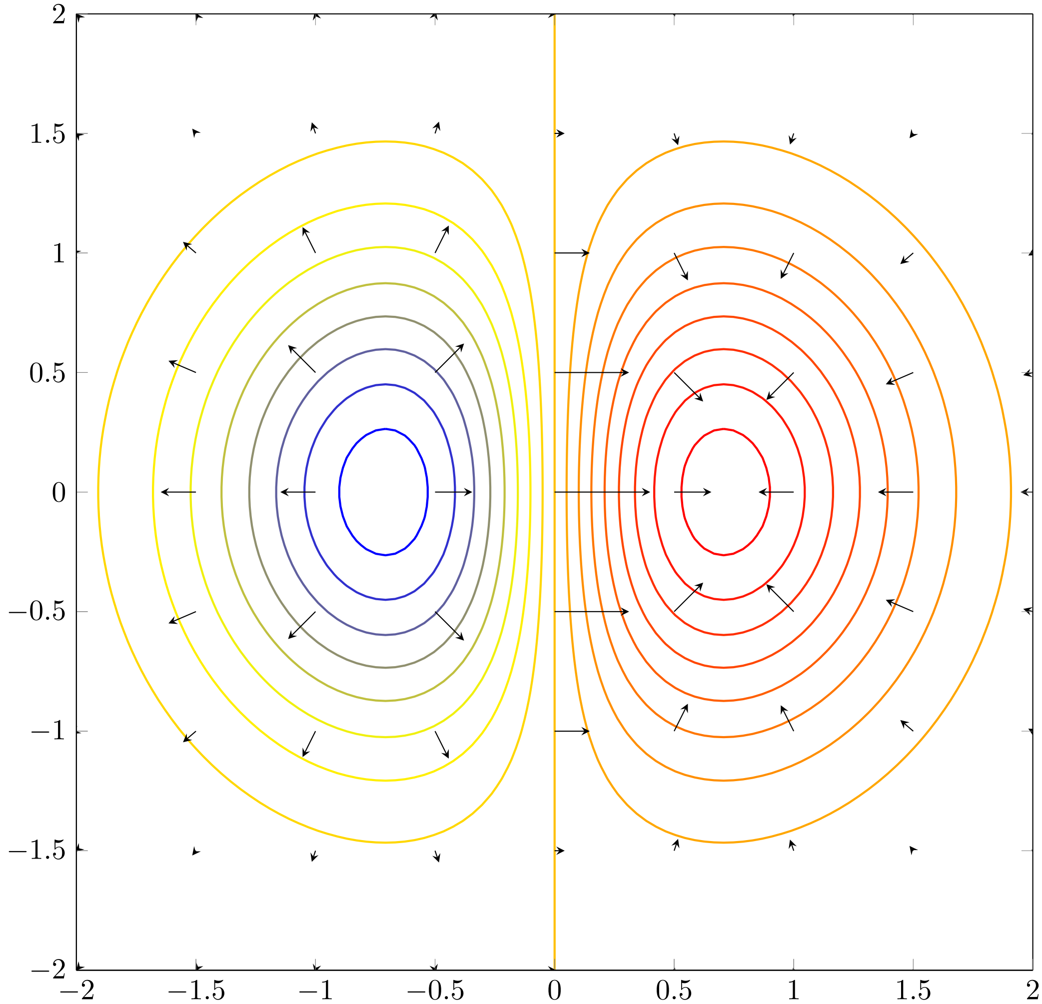
As always, be sure to show all the steps.

Solution: We compute

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v = u.$$

6. The plot shows the level curves of a differentiable function f with two critical points. Plot a gradient vector field of f that is consistent with the given level curves.

   Draw many vectors, making sure you have a variety of locations, magnitudes, and directions represented.



Solution: We know that the function is steeper when the level curves are closer together, so we should draw our arrows bigger when the level curves are closer together.

We also know that the gradient vector field is perpendicular to the level curves, so we want to make sure to draw our arrows perpendicular to the curves. There are two perpendicular directions, and we are not given enough information to decide which is the uphill direction and which is the downhill direction. Either choice is valid, but we must make the same choice throughout; our arrows cannot suddenly change direction.

The plot shown above assumes that the critical point on the left is a minimum and the critical point on the right is a maximum. With the opposite choice, we'd have to reverse the direction of all of the arrows. But they can't both be maxima, or else the vector field would have to change direction as it crosses the y-axis, which means it would have to have additional critical points there.

7. As usual, let $\mathbf{r} = \langle x, y, z \rangle$. Consider the force field

$$\mathbf{F}(\mathbf{r}) = K \frac{\mathbf{r}}{|\mathbf{r}|^3},$$

where K is a constant. If K is negative, this field might represent the force of gravity that a planet feels due to the sun at the origin. If K has either sign, this field might represent the force felt by a statically charged bit of dust due to a big charge at the origin.

Compute the work done on the particle if it moves in a straight line from $(2, 0, 0)$ to $(2, 1, 2)$.

Solution: We begin by parametrizing the line segment

$$x = 2, \qquad y = t, \qquad z = 2t, \qquad 0 \le t \le 1.$$

We compute

$$\begin{aligned} \mathbf{r} &= \langle 2, t, 2t \rangle, \\ \frac{d\mathbf{r}}{dt} &= \langle 0, 1, 2 \rangle. \end{aligned}$$

Letting C be the line segment, we then compute

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 K \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \langle 0, 1, 2 \rangle \, dt \\ &= K \int_0^1 (x^2 + y^2 + z^2)^{-3/2} \langle x, y, z \rangle \cdot \langle 0, 1, 2 \rangle \, dt \\ &= K \int_0^1 (4 + t^2 + 4t^2)^{-3/2} (y + 2z) \, dt \\ &= K \int_0^1 (5t^2 + 4)^{-3/2} (5t) \, dt \\ &= \frac{K}{2} \int_0^1 (5t^2 + 4)^{-3/2} \, d(5t^2 + 4) \\ &= -K \left[(5t^2 + 4)^{-1/2} \right]_0^1 \\ &= -K \left(\frac{1}{3} - \frac{1}{2} \right) \\ &= \frac{K}{6}. \end{aligned}$$

We can check our work if we learned the formula for graviational potential energy in physics class, from which we can conclude that $\mathbf{F}$ is the gradient vector field for $f(\mathbf{r}) = -\frac{K}{|\mathbf{r}|}$. Using the fundamental theorem for line integrals, we then conclude that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, 1, 2) - f(2, 0, 0) = -K \left(\frac{1}{\sqrt{2^2 + 1^2 + 2^2}} - \frac{1}{\sqrt{2^2 + 0^2 + 0^2}} \right) = \frac{K}{6},$$

as before.

8. Let

$$\mathbf{F} = \left\langle \frac{2x}{1+x^2+2y^2}, \frac{4y}{1+x^2+2y^2} \right\rangle.$$

(a) Verify that $\mathbf{F}$ is a conservative vector field.

Hint: Once you're done, read through your reasoning again. Does your reasoning work for the vector field from exercise 16.3.35? If so, your reasoning is bad, because the vector field from exercise 16.3.35 is not conservative.

Solution: Using the standard notation $\mathbf{F} = \langle P, Q \rangle$, we compute

$$P_y = -\frac{2x}{(1+x^2+2y^2)^2}(4y) = -\frac{8xy}{(1+x^2+2y^2)^2},$$
$$Q_x = -\frac{4y}{(1+x^2+2y^2)^2}(2x) = -\frac{8xy}{(1+x^2+2y^2)^2}.$$

Thus, $P_y = Q_x$.

We would like to conclude that $\mathbf{F}$ is conservative, but, as we saw in exercise 16.3.35, having $P_y = Q_x$ is not enough. We must also know that the domain is simply connected. In this case, we note that the denominator $1 + x^2 + 2y^2$ is never zero, so our domain is all of $\mathbb{R}^2$, which is certainly simply connected. Thus, Theorem 6 works, and so $\mathbf{F}$ is conservative.

(b) Let C_1 be the curve from $(1,0)$ to $(-1,0)$ going along the unit circle counterclockwise, and let C_2 be the curve from $(1,0)$ to $(-1,0)$ going along the unit circle clockwise. Determine which of $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is larger.

Hint: Are you sure you want to do all that work?

Solution: This is a trick question. By Theorem 2, if $\mathbf{F}$ is a conservative vector field, then the value of the line integral only depends on the endpoints of the path, not the path itself. Both C_1 and C_2 have the same endpoints, so the two line integrals are equal.

9. Let C be the circle of radius 3 centered at the origin, oriented counterclockwise. Compute

$$\int_C (1 - y^3)\, dx + (x^3 + e^{y^3})\, dy.$$

Solution: We note that C is the boundary of the disk of radius 3 centered at the origin, which we will denote by D. Then, using Green's Theorem, we know that

$$\begin{aligned}
\int_C (1 - y^3)\, dx + (x^3 + e^{y^3})\, dy &= \iint\limits_D \left(\frac{\partial}{\partial x} \left(x^3 + e^{y^3} \right) - \frac{\partial}{\partial y} \left(1 - y^3\right) \right) dA \\
&= \iint\limits_D \left(3x^2 - (-3y^2) \right) dA \\
&= \int_0^{2\pi} \int_0^3 (3r^2)\, r\, dr\, d\theta \\
&= 2\pi \left[\frac{3}{4} r^4 \right]_{r=0}^3 \\
&= \frac{243\pi}{2}.
\end{aligned}$$

10. Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field, and let $\mathbf{v} = \langle a, b, c \rangle$ be a constant vector. Verify that

$$\operatorname{div}(\mathbf{F} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{F}.$$

Solution: We begin by computing the left-hand side. First, from the formula for the cross product, we know that

$$\mathbf{F} \times \mathbf{v} = \langle Qc - Rb, Ra - Pc, Pb - Qa \rangle.$$

Taking the divergence and using the fact that $\mathbf{v}$ is constant, we obtain

$$\begin{aligned} \operatorname{div}(\mathbf{F} \times \mathbf{v}) &= \frac{\partial}{\partial x}(Qc - Rb) + \frac{\partial}{\partial y}(Ra - Pc) + \frac{\partial}{\partial z}(Pb - Qa) \\ &= Q_x c - R_x b + R_y a - P_y c + P_z b - Q_z a. \end{aligned}$$

Next, we work on the right-hand side. First, from the formula for curl, we know that

$$\operatorname{curl} \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

Taking the dot product with $\mathbf{v}$, we obtain

$$\begin{aligned} \mathbf{v} \cdot \operatorname{curl} \mathbf{F} &= a(R_y - Q_z) + b(P_z - R_x) + c(Q_x - P_y) \\ &= aR_y - aQ_z + bP_z - bR_x + cQ_x - cP_y. \end{aligned}$$

We can verify that each of these six terms is equal to a corresponding term that we computed for the left-hand side, so we conclude that

$$\operatorname{div}(\mathbf{F} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{F}.$$

11. Consider the surface defined by the parametric equations

$$x = 2\sin u \cos v, \qquad y = 3\sin u \sin v, \qquad z = 4\cos u, \qquad 0 \le u \le \pi, \qquad 0 \le v \le 2\pi.$$

(a) Write down an equation of the surface in terms of the variables x, y, and z only. Hints: Look at Example 4 in Section 16.6, and look at Section 12.6.

Solution: Based on the parametric equations of a sphere from Example 4 and our knowledge of how parametric equations work, it looks like we took a unit sphere and stretched in the x direction by 2, in the y direction by 3, and in the z direction by 4. Doing so gets us an ellipsoid. Based on its dimensions and our understanding of Section 12.6, or based on our understanding of how stretching shapes affect their equations, we obtain the following equation for the ellipsoid.

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{4}\right)^2 = 1. \tag{1}$$

Equivalently,

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

(b) Convince an imaginary skeptical peer that your answer to (a) is correct by plugging in the parametric equations into your answer for (a).

Solution: We plug our parametric equations into the left-hand side of (1) and hope we get 1 when we simplify. We compute

$$\begin{aligned}\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{4}\right)^2 &= (\sin u \cos v)^2 + (\sin u \sin v)^2 + \cos^2 u \\ &= \sin^2 u(\cos^2 v + \sin^2 v) + \cos^2 u \\ &= \sin^2 u + \cos^2 u \\ &= 1,\end{aligned}$$

as desired.

(c) Identify the surface.

Solution: The surface is an ellipsoid.

(d) Set up an integral for the surface area of the surface. Do not compute antiderivatives, but do as much work as you can up to that point. (As with most integrals, there isn't a formula for the answer, so don't try to evaluate the integral.)

Solution: As usual, we let $\mathbf{r} = \langle x, y, z\rangle$. Since we can use the textbook, we let

Example 10 be our guide as we compute

$$\begin{aligned}
\mathbf{r}_u &= \langle 2\cos u\cos v, 3\cos u\sin v, -4\sin u\rangle \\
\mathbf{r}_v &= \langle -2\sin u\sin v, 3\sin u\cos v, 0\rangle \\
\mathbf{r}_u\times\mathbf{r}_v &= \langle 12\sin^2 u\cos v, 8\sin^2 u\sin v, 6\sin u\cos u\cos^2 v + 6\sin u\cos u\sin^2 v\rangle \\
&= \langle 12\sin^2 u\cos v, 8\sin^2 u\sin v, 6\sin u\cos u\rangle.
\end{aligned}$$

Thus, the area of the surface is

$$\begin{aligned}
A &= \iint\limits_D |\mathbf{r}_u\times\mathbf{r}_v|\ dA \\
&= \int_0^\pi\int_0^{2\pi}\sqrt{144\sin^4 u\cos^2 v + 64\sin^4 u\sin^2 v + 36\sin^2 u\cos^2 u}\,dv\,du \\
&= \int_0^\pi\int_0^{2\pi}\sin u\sqrt{144\sin^2 u\cos^2 v + 64\sin^2 u\sin^2 v + 36\cos^2 u}\,dv\,du.
\end{aligned}$$

Unlike in Example 10, because the coefficients of the trigonometric functions are not equal to each other, there is not really a way to simplify this expression further. There are equivalent expressions that are equally complicated, but no expressions that are simpler.

12. Let S be the sphere of radius 2 centered around the origin. Compute

$$\iint\limits_S e^{-\frac{x^2+y^2+z^2}{4}}\,dS.$$

Hint: Are you sure you want to do all that work?

Solution: The sphere has equation $x^2 + y^2 + z^2 = 4$. Since we are integrating over this sphere, we can simplify the integrand, obtaining

$$\iint\limits_S e^{-\frac{4}{4}}\,dS = \iint\limits_S \frac{1}{e}\,dS$$

Since $\frac{1}{e}$ is a constant, the answer is just $\frac{1}{e}$ times the area of the sphere. The area of a sphere of radius 2 is $4\pi(2)^2 = 16\pi$. We conclude that

$$\iint\limits_S e^{-\frac{x^2+y^2+z^2}{4}}\,dS = \frac{16\pi}{e}.$$

13. Let $\mathbf{F} = -y\mathbf{i} + \left(x + e^{-y^2}\right)\mathbf{j}$.

Hints: Don't do things the hard way.

(a) Let D be the disk $x^2 + y^2 \leq 4$, oriented upwards. Compute $\iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Solution: We compute

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial}{\partial x}\left(x + e^{-y^2}\right) - \frac{\partial}{\partial y}(-y)\right)\mathbf{k} = 2\mathbf{k}.$$

It's clear that $\operatorname{curl} \mathbf{F}$ is a simpler expression than $\mathbf{F}$, so we should compute $\iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ directly. If we used Stokes' Theorem, we'd have to compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, which involves e^{-y^2}, so it would be harder.

The unit normal vector to D is $\mathbf{n} = \mathbf{k}$. Thus,

$$\iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D 2\mathbf{k} \cdot \mathbf{k}\, dS = \iint_D 2\, dS = 2A(D) = 2\pi(2)^2 = 8\pi.$$

(b) Let H be the hemisphere $x^2+y^2+z^2 = 4$, $z \geq 0$, oriented upwards. Compute $\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Solution: The boundary of H is the circle $x^2 + y^2 = 4$. This boundary is the same as the boundary of D, with the same counterclockwise-from-above orientation because both H and D are oriented upwards. Letting C denote this circle, by Stokes's Theorem, we have

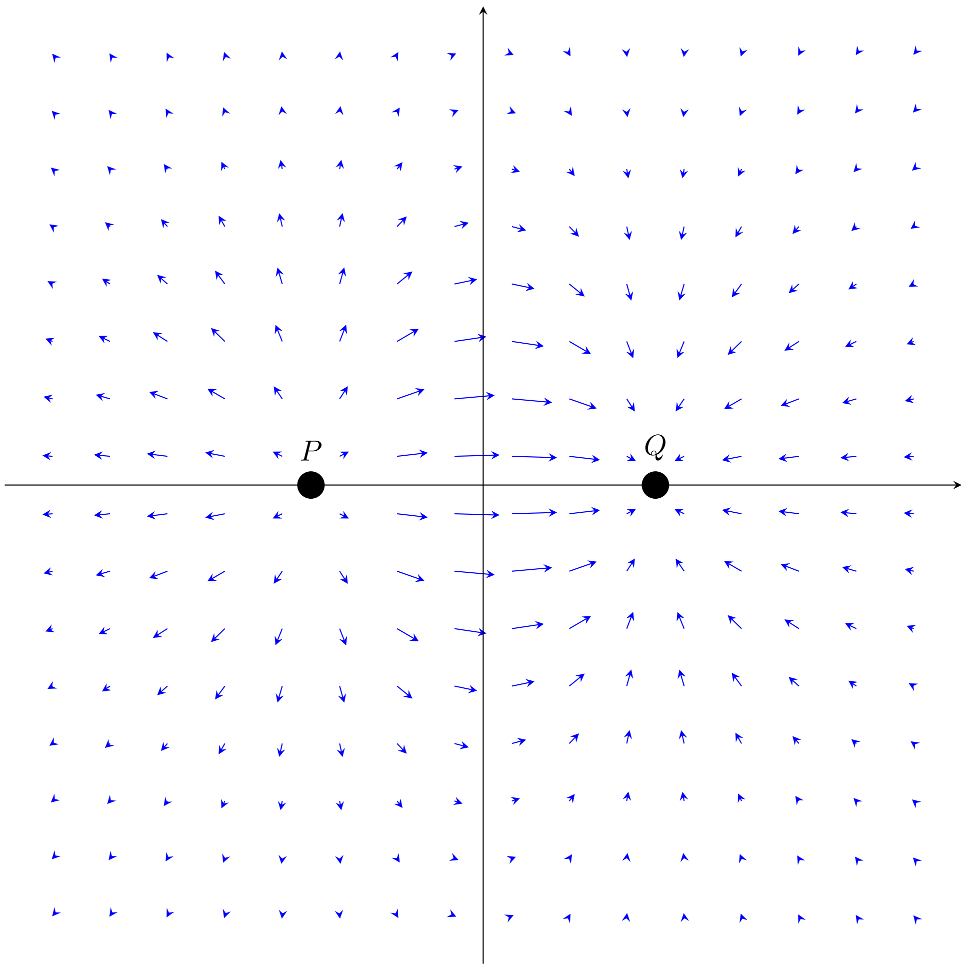
$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 8\pi.$$

(c) Let P be the part of the paraboloid $x^2 + y^2 + z = 4$ with $z \geq 0$, oriented *downwards*. Compute $\iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Solution: Once again, the boundary of P is the circle $x^2 + y^2 = 4$, but this time it is oriented clockwise when viewed from above because P is oriented downwards. Using the above notation, the boundary of P is $-C$, so by Stokes's Theorem,

$$\iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{-C} \mathbf{F} \cdot d\mathbf{r} = -\oint_C \mathbf{F} \cdot d\mathbf{r} = -\iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = -8\pi.$$

14. Consider the vector field **F** drawn below.



Draw two points on the vector field so that div **F** is positive at one of the points and negative at the other point.

As always, make sure to justify your choices, and make sure to make it clear which point is which. I recommend labels.

Solution: If **F** represents the motion of a gas or a cloud of dust or whatever, then div **F** represents creation or expansion. Near P, all the vectors are pointing away from P, so the gas is expanding or being created there, and so div **F** is positive at P.

By the same reasoning, near Q, all the vectors are pointing towards Q, so the gas is contracting or being destroyed there, and so div **F** is negative at Q.

As an aside, this vector field is the answer to question 6. Interpreting the function f as elevation and flipping a sign, we can interpret what's going on here as marbles rolling down a hill. They *spread out* from the mountaintop, and *come together* at the crater.