Math 253A Final

May 14, 2020

- Write your solutions and upload them on Gradescope, just like your homework assignments. You can write your solutions on the exam pages or on separate sheets of paper, your choice.
- Only use the resources allowed on the exam honor code certification form.
- Be sure to include the exam honor code certification form with your solutions. If you are unable to print it, copy the form by hand.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.

1. Consider the part of the surface $z = 1 + x^2y^2$ that lies above the disk $x^2 + y^2 \leq 4$.

   Set up a double integral in polar coordinates to compute its area. Do not compute antiderivatives, but do as much work as you can up to that point.

2. Consider a brick with constant density ρ and side lengths a, b, and c. Place the brick so that its center of mass is at the origin, the sides of length a are parallel to the x-axis, the sides of length b are parallel to the y-axis, and the sides of length c are parallel to the z-axis.

   Find the moments of inertia of the brick about each of the coordinate axes.

3. Consider the triple integral

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} z\,dz\,dx\,dy.$$

(a) Describe the domain of integration. Be specific. Use words or pictures to describe the shape qualitatively, and use numbers to describe the shape quantitatively.

(b) Express this integral in cylindrical coordiates.

(c) Evaluate the integral.

4. Evaluate

$$\iiint\limits_E y^2\, dV,$$

where E is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$.

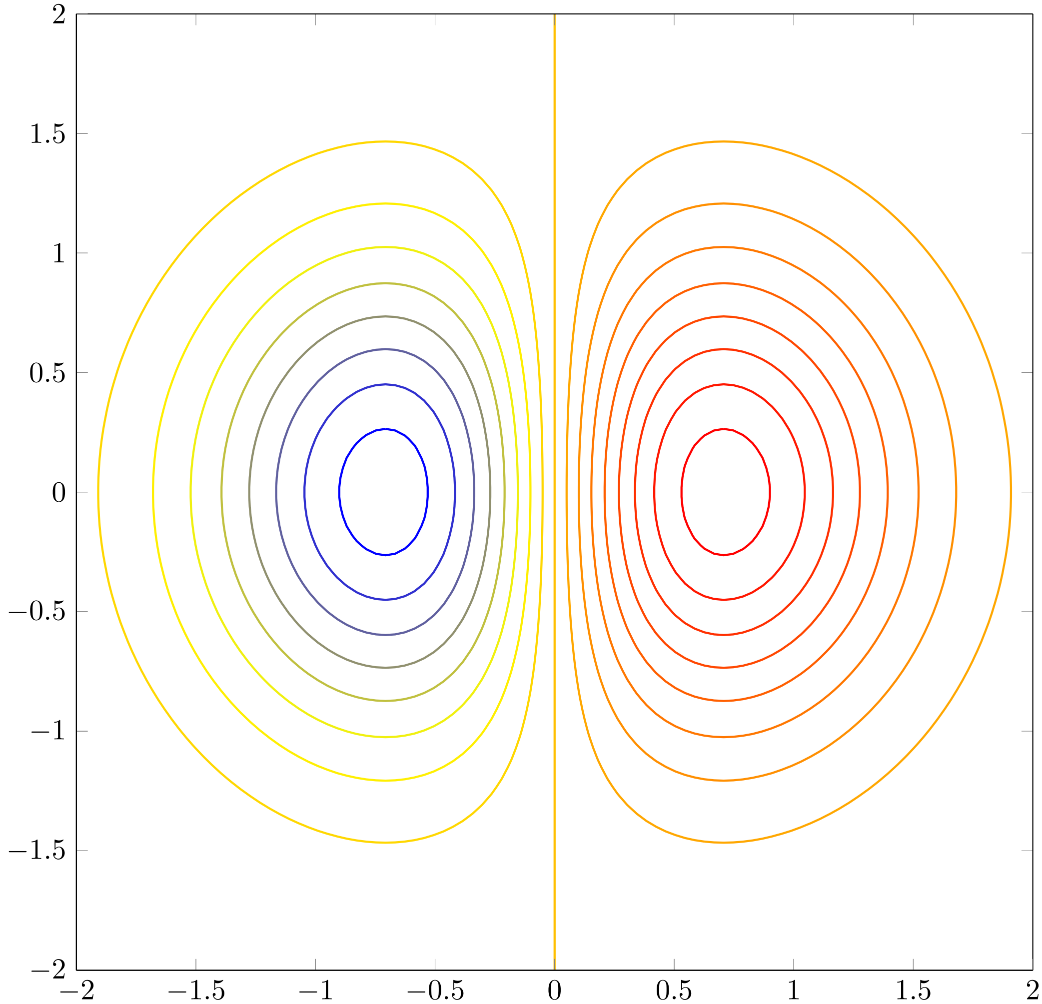
5. Compute the Jacobian of the polar coordinate transformation

$$x = u\cos v, \qquad\qquad y = u\sin v.$$

As always, be sure to show all the steps.

6. The plot shows the level curves of a differentiable function f with two critical points. Plot a gradient vector field of f that is consistent with the given level curves.

   Draw many vectors, making sure you have a variety of locations, magnitudes, and directions represented.



7. As usual, let $\mathbf{r} = \langle x, y, z \rangle$. Consider the force field

$$\mathbf{F}(\mathbf{r}) = K\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

where K is a constant. If K is negative, this field might represent the force of gravity that a planet feels due to the sun at the origin. If K has either sign, this field might represent the force felt by a statically charged bit of dust due to a big charge at the origin.

Compute the work done on the particle if it moves in a straight line from $(2, 0, 0)$ to $(2, 1, 2)$.

8. Let

$$\mathbf{F} = \left\langle \frac{2x}{1+x^2+2y^2}, \frac{4y}{1+x^2+2y^2} \right\rangle.$$

(a) Verify that **F** is a conservative vector field.

Hint: Once you're done, read through your reasoning again. Does your reasoning work for the vector field from exercise 16.3.35? If so, your reasoning is bad, because the vector field from exercise 16.3.35 is not conservative.

(b) Let C_1 be the curve from $(1,0)$ to $(-1,0)$ going along the unit circle counterclockwise, and let C_2 be the curve from $(1,0)$ to $(-1,0)$ going along the unit circle clockwise. Determine which of $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is larger.

Hint: Are you sure you want to do all that work?

9. Let C be the circle of radius 3 centered at the origin, oriented counterclockwise. Compute

$$\int_C (1 - y^3)\,dx + (x^3 + e^{y^3})\,dy.$$

10. Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field, and let $\mathbf{v} = \langle a, b, c \rangle$ be a constant vector. Verify that

$$\operatorname{div}(\mathbf{F} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{F}.$$

11. Consider the surface defined by the parametric equations

$$x = 2\sin u\cos v, \qquad y = 3\sin u\sin v, \qquad z = 4\cos u, \qquad 0 \le u \le \pi, \qquad 0 \le v \le 2\pi.$$

(a) Write down an equation of the surface in terms of the variables x, y, and z only.
Hints: Look at Example 4 in Section 16.6, and look at Section 12.6.

(b) Convince an imaginary skeptical peer that your answer to (a) is correct by plugging in the parametric equations into your answer for (a).

(c) Identify the surface.

(d) Set up an integral for the surface area of the surface. Do not compute antiderivatives, but do as much work as you can up to that point. (As with most integrals, there isn't a formula for the answer, so don't try to evaluate the integral.)

12. Let S be the sphere of radius 2 centered around the origin. Compute

$$\iint\limits_S e^{-\frac{x^2+y^2+z^2}{4}}\, dS.$$

Hint: Are you sure you want to do all that work?

13. Let $\mathbf{F} = -y\mathbf{i} + \left(x + e^{-y^2}\right)\mathbf{j}$.

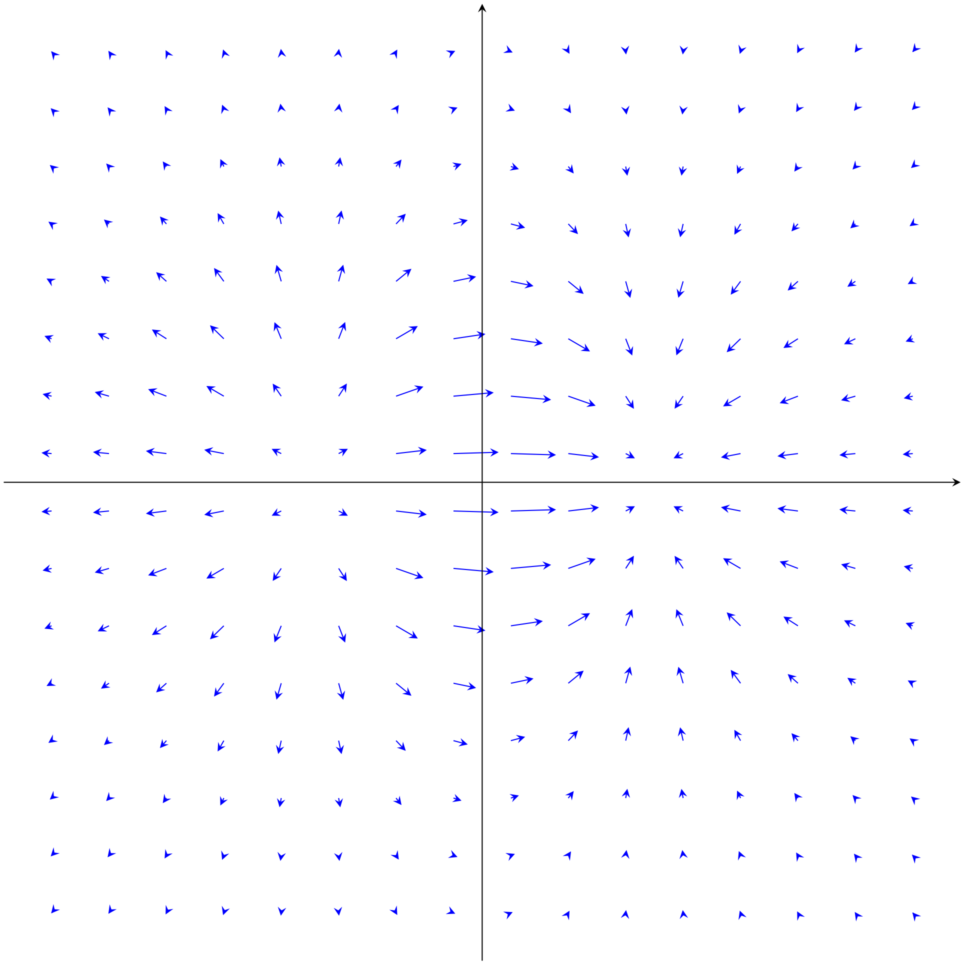
Hints: Don't do things the hard way.

(a) Let D be the disk $x^2 + y^2 \leq 4$, oriented upwards. Compute $\iint_D \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

(b) Let H be the hemisphere $x^2+y^2+z^2 = 4$, $z \geq 0$, oriented upwards. Compute $\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

(c) Let P be the part of the paraboloid $x^2 + y^2 + z = 4$ with $z \geq 0$, oriented *downwards*. Compute $\iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

14. Consider the vector field **F** drawn below.



Draw two points on the vector field so that div **F** is positive at one of the points and negative at the other point.

As always, make sure to justify your choices, and make sure to make it clear which point is which. I recommend labels.