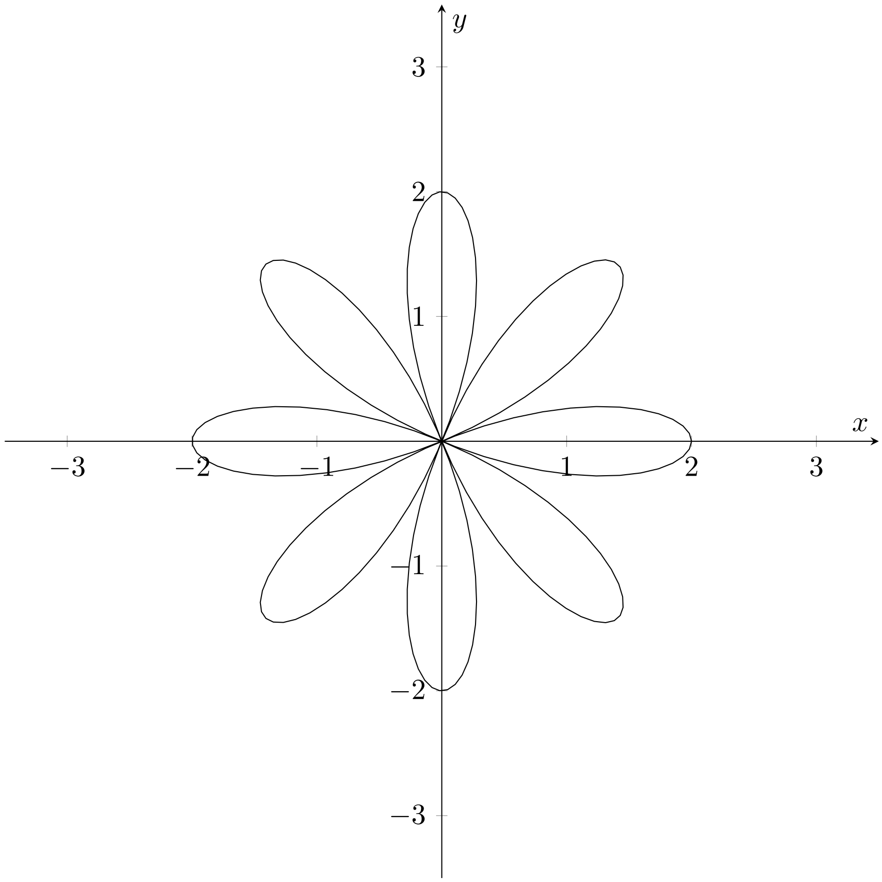
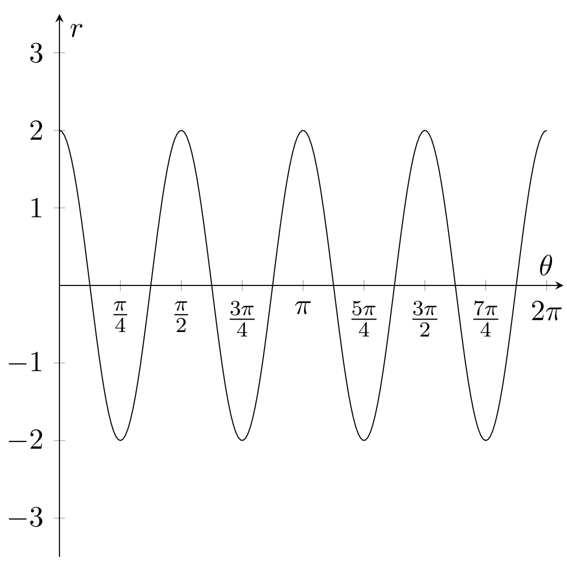
Name: ____________________ ID: ____________

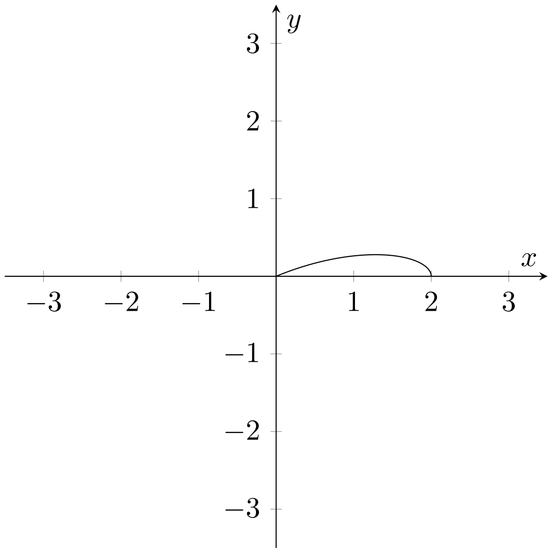
1. Sketch the polar curve $r = 2\cos 4\theta$.



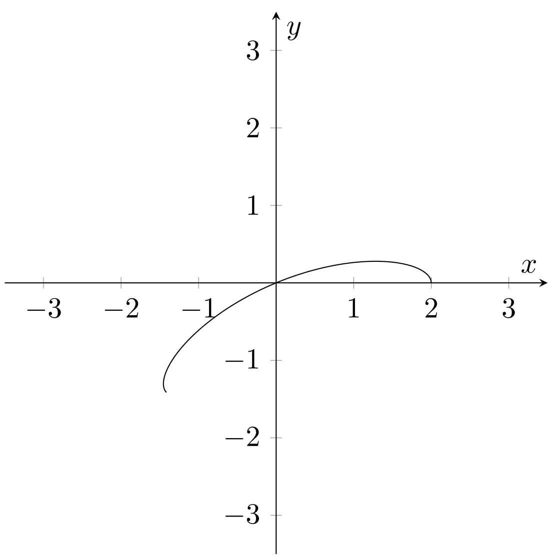
Solution: We begin with our helper Cartesian plot of $r = \cos 4\theta$.



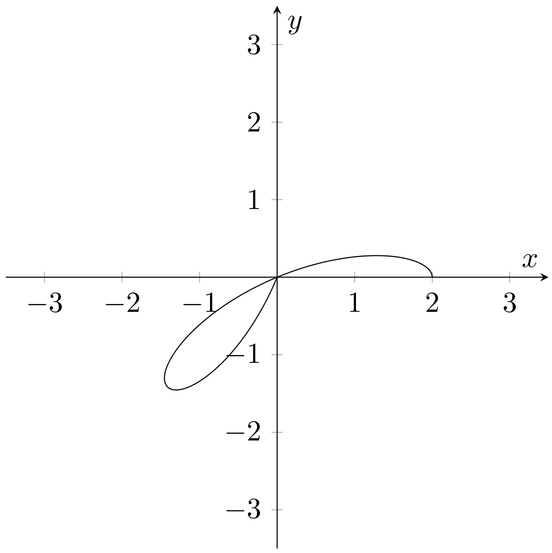
We see that as θ goes from 0 to $\frac{\pi}{8}$, r goes from 2 to 0. We draw the corresponding part of the curve in the region $0 \leq \theta \leq \frac{\pi}{8}$.



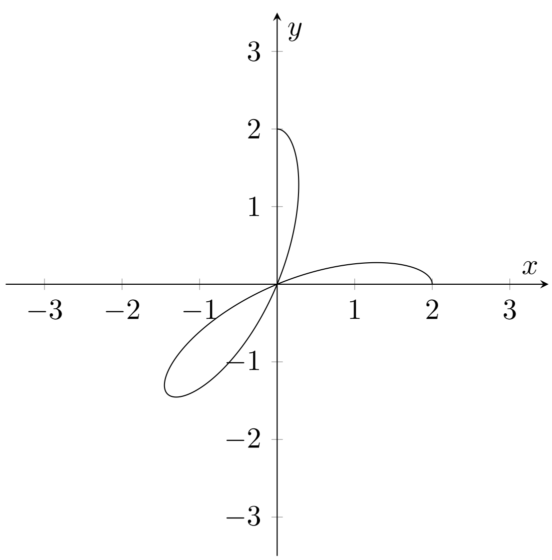
Then, as θ goes from $\frac{\pi}{8}$ to $\frac{\pi}{4}$, r goes from 0 to -2. Because r is negative, we draw the next part of the curve in the region opposite the region with $\frac{\pi}{8} \leq \theta \leq \frac{\pi}{4}$, namely the region with $\frac{9\pi}{8} \leq \theta \leq \frac{3\pi}{4}$. Our plot now looks like



Then, as θ goes from $\frac{\pi}{4}$ to $\frac{3\pi}{8}$, r goes from -2 to 0. Because r is negative, we draw the next part of the curve in the region opposite the region with $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{8}$, namely the region with $\frac{3\pi}{4} \leq \theta \leq \frac{11\pi}{8}$. Our plot now looks like



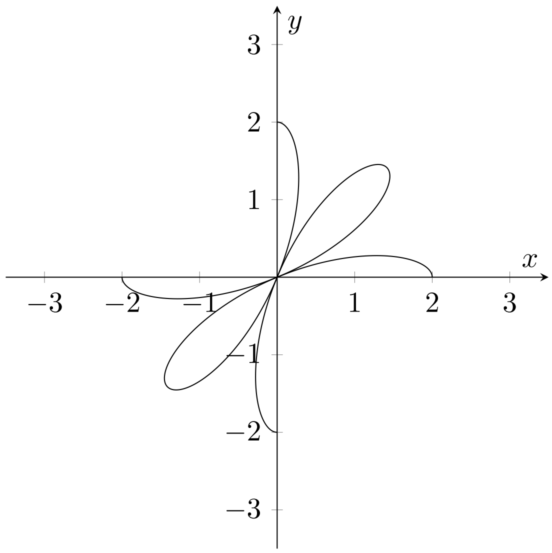
Next, as θ goes from $\frac{3\pi}{8}$ to $\frac{\pi}{2}$, r goes from 0 to 2. We continue our plot in the region $\frac{3\pi}{8} \leq \theta\frac{\pi}{2}$.



We note that our equation does not change when we replace θ by $\theta + \pi$. Indeed,

$$2\cos(4(\theta + \pi)) = 2\cos(4\theta + 4\pi) = 2\cos 4\theta.$$

Thus, our curve has 180° rotational symmetry. Rotating what we've drawn by 180°, our plot is now

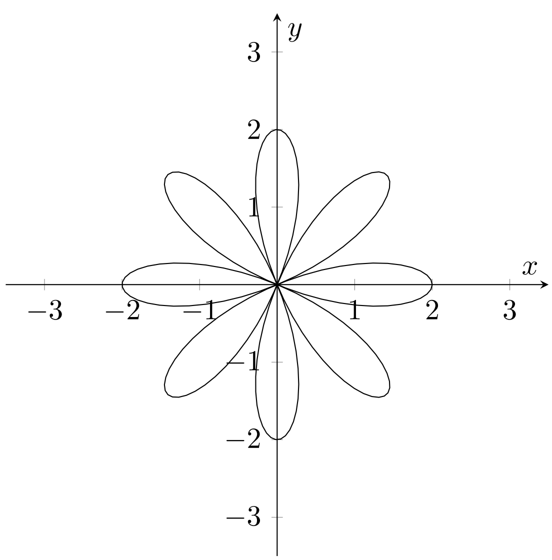


In fact, we could have saved ourselves some work by noticing that the equation does not change when we replace θ by $\theta + \frac{\pi}{2}$, so our curve is actually invariant under 90° rotations also.

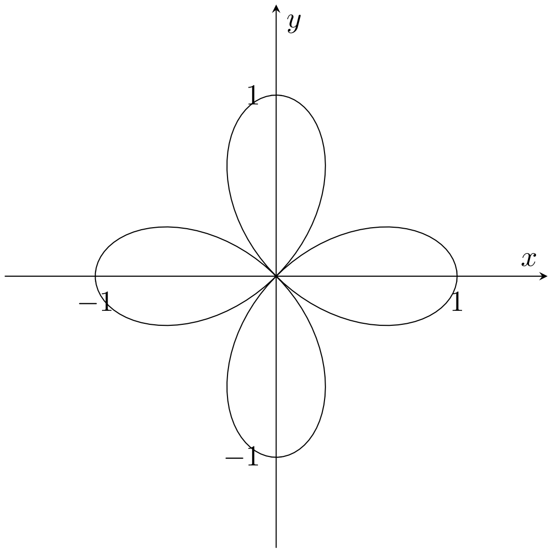
Regardless, we can then notice that our equation does not change when we replace θ by $-\theta$. Indeed

$$2\cos(4(-\theta)) = 2\cos(-4\theta) = 2\cos 4\theta$$

because cosine is even. Therefore, our curve has reflectional symmetry across the x-axis. We reflect what we've drawn so far and add it to our plot, obtaining our answer.



2. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



Solution: The formula for the area enclosed by a polar curve is

$$A = \int \frac{1}{2} r^2 \, d\theta.$$

However, we need to know the bounds of this integral. A loop of the four-leaved rose begins and ends at the origin, that is, when $r = 0$. We need to find the corresponding values of θ. Setting $r = 0$, we find that $\cos 2\theta = 0$, so $2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, and so $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$. Thus, one choice is to consider $\frac{\pi}{4} \leq \theta \frac{3\pi}{4}$. This part of the curve describes the bottom loop of the rose (not the top one, because r is negative). Another reasonable choice is to consider $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. We compute

$$\begin{aligned} A &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} r^2 \, d\theta \\ &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2 2\theta \, d\theta. \end{aligned}$$

At this point, we can use the fact that we're covering a full period of $\cos^2 2\theta$ as θ goes from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, so its average value is $\frac{1}{2}$. Computing further, we find that

$$A = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} \, d\theta = \frac{1}{2} \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) \frac{1}{2} = \frac{\pi}{8}.$$

Alternatively, we recall that $\cos^2\theta + \sin^2\theta = 1$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$. Adding the two equations, we obtain $2\cos^2\theta = 1 + \cos 2\theta$. Substituting 2θ for θ, we find that $2\cos^2 2\theta = 1 + \cos 4\theta$, which gives $\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$. Thus, we can instead continue our computation with

$$
\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1+\cos 4\theta}{2}\,d\theta \\
&= \frac{1}{4}\left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1\,d\theta + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos 4\theta\,d\theta\right) \\
&= \frac{1}{4}\left(\theta\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \frac{1}{4}\sin 4\theta\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\right) \\
&= \frac{1}{4}\left(\frac{3\pi}{4} - \frac{\pi}{4} + 0 - 0\right) = \frac{\pi}{8}.
\end{aligned}
$$