Name: ______________________________ ID: ________________

1. Find the exact length of the curve

$$36y^2 = \left(x^2 - 4\right)^3, \qquad 2 \le x \le 3, \qquad y \ge 0.$$

Solution: Using $y \ge 0$, we isolate y, obtaining

$$y = \tfrac{1}{6}\left(x^2 - 4\right)^{3/2}.$$

Next, we compute

$$\frac{dy}{dx} = \tfrac{1}{6}\tfrac{3}{2}\left(x^2 - 4\right)^{1/2}(2x) = \tfrac{1}{2}x\left(x^2 - 4\right)^{1/2}.$$

Thus,

$$\begin{aligned}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \tfrac{1}{4}x^2(x^2 - 4)}, \\ &= \sqrt{\tfrac{1}{4}x^4 - x^2 + 1}, \\ &= \sqrt{\left(\tfrac{1}{2}x^2 - 1\right)^2}, \\ &= \tfrac{1}{2}x^2 - 1\end{aligned}$$

because $\frac{1}{2}x^2 - 1 \ge 0$ when $2 \le x \le 3$. Finally, using the arclength formula, we compute that

$$\begin{aligned}L = \int_2^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_2^3 \left(\tfrac{1}{2}x^2 - 1\right)\,dx &= \tfrac{1}{6}x^3\Big|_2^3 - x\Big|_2^3 \\ &= \tfrac{1}{6}(27 - 8) - (3 - 2) = \tfrac{19}{6} - 1 = \tfrac{13}{6}.\end{aligned}$$

2. Consider the cycloid

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta).$$

Find the slope of the tangent line to the cycloid at the point where $\theta = \frac{\pi}{3}$.

Solution: We compute that

$$\frac{dx}{d\theta} = r(1 - \cos\theta), \qquad \frac{dy}{d\theta} = r\sin\theta.$$

Thus, when $\theta = \frac{\pi}{3}$, we have

$$\frac{dx}{d\theta} = r(1 - \tfrac{1}{2}) = \tfrac{r}{2}, \qquad \frac{dy}{d\theta} = r\tfrac{\sqrt{3}}{2}.$$

Thus, the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{r\frac{\sqrt{3}}{2}}{\frac{r}{2}} = \sqrt{3}.$$