Math 243 Midterm 2

April 9, 2020

- Write your solutions and upload them on Gradescope, just like your homework assignments. You can write your solutions on the exam pages or on separate sheets of paper, your choice.
- Only use the resources allowed on the exam honor code certification form.
- Be sure to include the exam honor code certification form with your solutions. If you are unable to print it, copy the form by hand.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.

1. Consider the triangle determined by the points $P(2,-1,0)$, $Q(4,1,1)$, $R(4,-5,5)$.

   (a) Find the lengths of the three sides of the triangle.

Solution: We compute

$$\overrightarrow{PQ} = \langle 2,2,1\rangle, \qquad \overrightarrow{PR} = \langle 2,-4,5\rangle, \qquad \overrightarrow{QR} = \langle 0,-6,4\rangle,$$
$$|PQ| = \sqrt{4+4+1} = 3, \quad |PR| = \sqrt{4+16+25} = 3\sqrt{5}, \quad |QR| = \sqrt{0+36+16} = 2\sqrt{13}.$$

   (b) Is it a right triangle?

Solution: We check if the side lengths satisfy the Pythagorean theorem.

$$|PQ|^2 + |PR|^2 = 9 + 45 = 54, \qquad |QR|^2 = 52.$$

Since $|PQ|^2 + |PR|^2 \neq |QR|^2$, the triangle is not right by the Pythagorean theorem. Alternatively, $\overrightarrow{PQ} \cdot \overrightarrow{PR} = 4 - 8 + 5 = 1$, so these two vectors are not perpendicular. Thus, the sides PQ and PR do not form a right angle.

   (c) Is it an isosceles triangle?

Solution: No two sides are the same length, so it is not an isosceles triangle.

2. Consider the lines $x + 2y = 7$ and $5x - y = 3$ in the plane. Find the acute angle between the lines.

Solution: We can write the first equation as $y = \frac{7}{2} - \frac{x}{2}$ and the second line as $y = 5x - 3$. The first line has slope $-\frac{1}{2}$ and the second line has slope 5. Because slope is rise over run, the vector $\langle 1, -\frac{1}{2} \rangle$ points in the direction of the first line, and the vector $\langle 1, 5 \rangle$ points in the direction of the second line. We compute that

$$\cos\theta = \frac{\langle 1, -\frac{1}{2}\rangle \cdot \langle 1, 5\rangle}{|\langle 1, -\frac{1}{2}\rangle|\,|\langle 1, 5\rangle|} = \frac{1 - \frac{5}{2}}{\sqrt{1 + \frac{1}{4}}\sqrt{1 + 25}} = \frac{-\frac{3}{2}}{\frac{\sqrt{5}}{2}\sqrt{26}} = -\frac{3}{\sqrt{130}}.$$

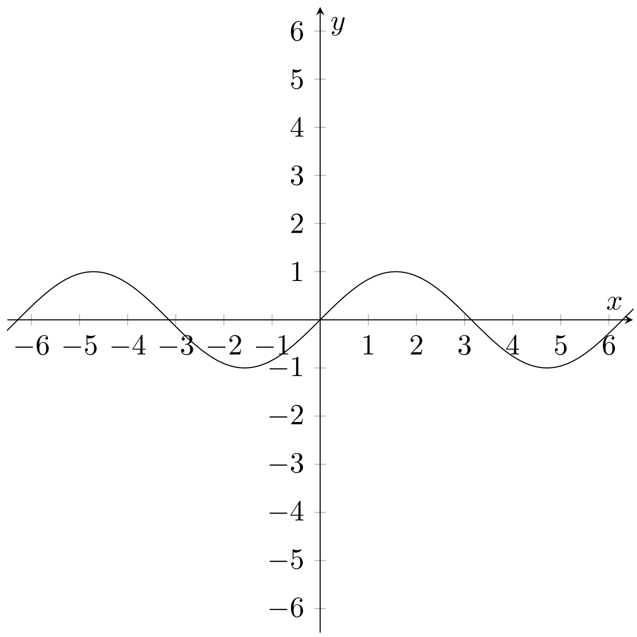
Using basic calculator features, we compute that the angle between the lines is $\theta = \cos^{-1}\left(-\frac{3}{\sqrt{130}}\right) \approx 105.3°$. Since the problem asks for the acute angle between these lines, our answer is $180° - \cos^{-1}\left(-\frac{3}{\sqrt{130}}\right) = \cos^{-1}\left(\frac{3}{\sqrt{130}}\right) \approx 74.7°$.

Alternatively, based on what we know about planes in three dimensions, we can read the coefficients of the line equations to see that $\mathbf{n}_1 = \langle 1, 2 \rangle$ is perpendicular to the first line, and $\mathbf{n}_2 = \langle 5, -1 \rangle$ is perpendicular to the second line. Thus, the angle between the lines is the angle between these normal vectors, which is given by

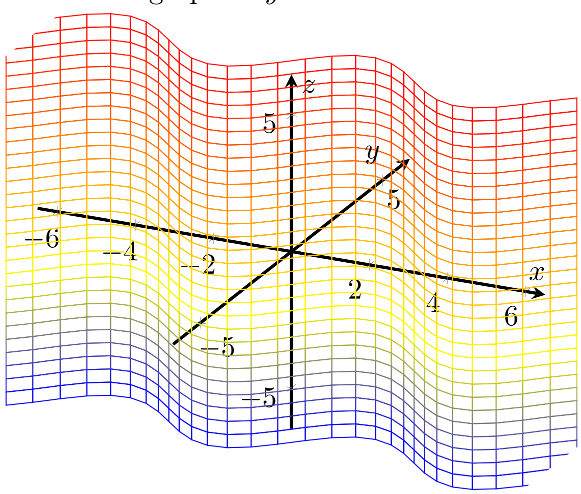
$$\cos\theta = \frac{\langle 1, 2\rangle \cdot \langle 5, -1\rangle}{|\langle 1, 2\rangle|\,|\langle 5, -1\rangle|} = \frac{5 - 2}{\sqrt{1^2 + 2^2}\sqrt{5^2 + 1^2}} = \frac{3}{\sqrt{5}\sqrt{26}} = \frac{3}{\sqrt{130}}$$

We once again find that the angle between the lines is $\cos^{-1}\left(\frac{3}{\sqrt{130}}\right) \approx 74.7°$.

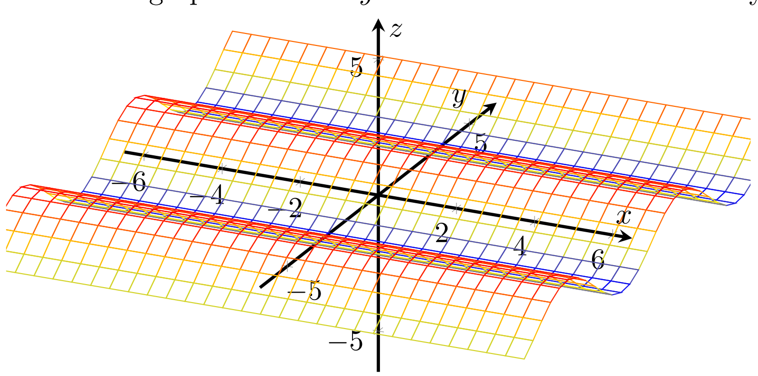
3. (a) Sketch the graph of $y = \sin x$ as a curve in $\mathbb{R}^2$.



(b) Sketch the graph of $y = \sin x$ as a surface in $\mathbb{R}^3$. Describe your sketch if needed.



(c) Sketch the graph of $z = \sin y$ as a surface in $\mathbb{R}^3$. Describe your sketch if needed.



4. Consider the curve $y = \sin x + \sin 2x$ in $\mathbb{R}^2$. Find its curvature in terms of x.

Solution: Using formula 11 in the textbook with $f(x) = \sin x + \sin 2x$, we compute

$$\begin{aligned} f'(x) &= \cos x + 2\cos 2x, \\ f''(x) &= -\sin x - 4\sin 2x, \\ \kappa(x) &= \frac{|\sin x + 4\sin 2x|}{(1 + (\cos x + 2\cos 2x)^2)^{3/2}}. \end{aligned}$$