Math 243 Midterm 1

February 20, 2020

Name: ______________________________ ID: ________________

- Each page has a space at the top for the last 4 digits of your student ID. Make sure that you fill that out on at least one side of every sheet of paper.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- Simplify your answers as much as possible.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.
- You may not use any tools or resources other than writing implements. In particular, no calculators, phones, notes, and so forth.

1. Set up an integral that computes the arc length of the curve defined by

$$y = \sqrt[3]{x}, \qquad\qquad 1 \le x \le 6.$$

You do not need to evaluate the integral, but be sure to simplify the integrand.

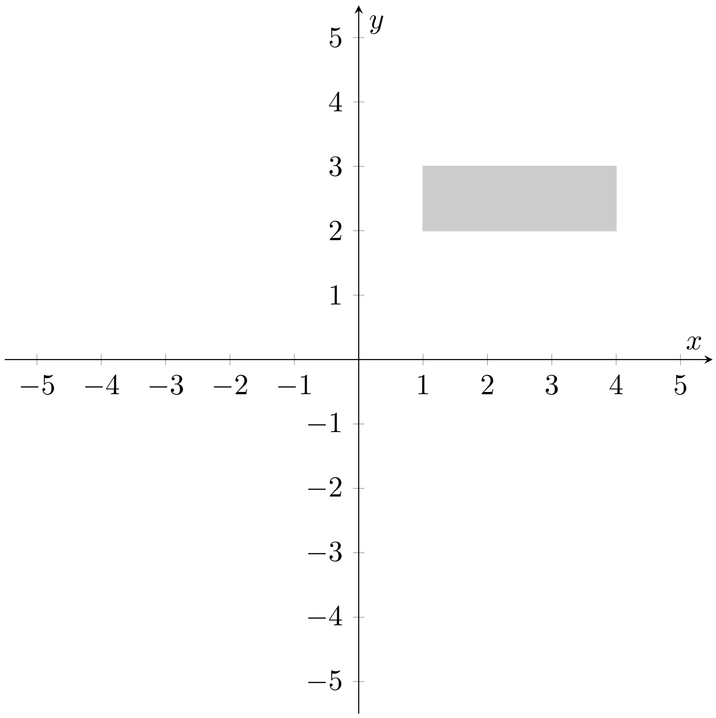
Solution: We compute

$$\begin{aligned} y &= x^{1/3}, \\ \frac{dy}{dx} &= \tfrac{1}{3}x^{-2/3}, \\ \left(\frac{dy}{dx}\right)^2 &= \tfrac{1}{9}x^{-4/3}, \\ L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt \\ &= \int_1^6 \sqrt{1 + \tfrac{1}{9}x^{-4/3}}\, dx. \end{aligned}$$

2. Suppose a curve is given by the parametric equations

$$x = f(t), \qquad\qquad y = g(t),$$

where the range of f is $[1, 4]$ and the range of g is $[2, 3]$. What can you say about the curve?



Solution: Because the x-values of points on the curve are between 1 and 4 and the y-values of points on the curve are between 2 and 3, the curve stays inside the rectangle drawn above.

3. Consider the curve parametrized by the equations

$$x = \frac{t}{1+t}, \qquad y = \sqrt{1+t}.$$

Compute $\frac{dy}{dx}$.

Solution: We compute

$$\frac{dx}{dt} = \frac{(1)(1+t) - (t)(1)}{(1+t)^2} = \frac{1}{(1+t)^2}, \qquad \frac{dy}{dt} = \frac{1}{2\sqrt{1+t}}.$$

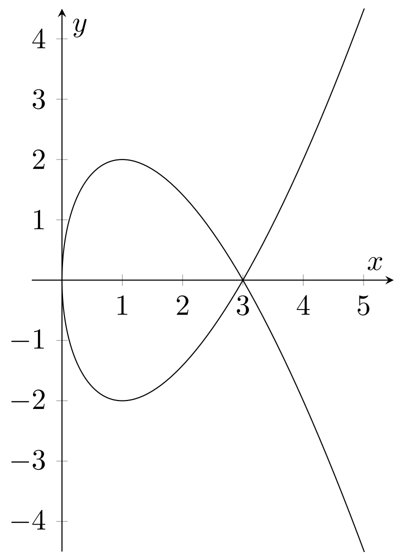
Thus,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2\sqrt{1+t}}}{\frac{1}{(1+t)^2}} = \frac{(1+t)^2}{2\sqrt{1+t}} = \frac{1}{2}\left(1+t\right)^{3/2}.$$

4. Consider the curve parametrized by the equations

$$x = t^2, \qquad y = t^3 - 3t.$$

Find the area inside the loop.



Solution: We compute that

$$\begin{aligned} A &= \int y\,dx \\ &= \int y\frac{dx}{dt}\,dt \\ &= \int (t^3 - 3t)(2t)\,dt \\ &= \int \left(2t^4 - 6t^2\right)\,dt \\ &= \left(\frac{2}{5}t^5 - 2t^3\right)\Big|_{t=?}^{?} \end{aligned}$$

We realize that we need to know the bounds for the integral. Since the curve is symmetric, it might be easier to do just the top half or just the bottom half of the loop. It looks like the loop goes from $x = 0$ to $x = 3$. From the equation $x = t^2$, we see that these correspond to t-values of $t = 0$ and $t = \pm\sqrt{3}$. We can check that these values are correct by plugging in $t = 0$ and $t = \pm\sqrt{3}$ into $y = t^3 - 3t$ and observing that we do indeed get $y = 0$. Regardless, we can go ahead and integrate from $t = 0$ to $t = \sqrt{3}$.

We obtain that the area of half the loop is

$$\begin{aligned}
&= \left(\frac{2}{5}t^5 - 2t^3\right)\Big|_{t=0}^{\sqrt{3}} \\
&= \frac{2}{5}9\sqrt{3} - 2(3\sqrt{3}) + 0 - 0 \\
&= \left(\frac{18}{5} - 6\right)\sqrt{3} \\
&= -\frac{12\sqrt{3}}{5}.
\end{aligned}$$

We mysteriously obtained a negative number, but with some thought we can see why that happened. When $0 < t < \sqrt{3}$, we have that $y = t^3 - 3t$ is negative, so the curve is below the x-axis. Thus, we expect to get the negative of the area when we compute the integral. Thus, the area of half the loop is is $\frac{12\sqrt{3}}{5}$, so the area of the full loop is $\frac{24\sqrt{3}}{5}$.

Alternatively, we can eliminate the parameter. We see that $x = t^2$ gives $t = \pm\sqrt{x}$. We can pick $t = \sqrt{x}$, though we'll only get half the curve that way. Then $y = x^{3/2} - 3x^{1/2}$. This time, we're integrating with respect to x, and our bounds for x on the loop are $0 \le x \le 3$. We compute that the area of half the loop is

$$\int_0^3 \left(x^{3/2} - 3x^{1/2}\right)\, dx = \left(\frac{2}{5}x^{5/2} - (3)\frac{2}{3}x^{3/2}\right)\Big|_{x=0}^{x=3} = \frac{2}{5}9\sqrt{3} - 2(3\sqrt{3}) + 0 - 0 = -\frac{12\sqrt{3}}{5}$$

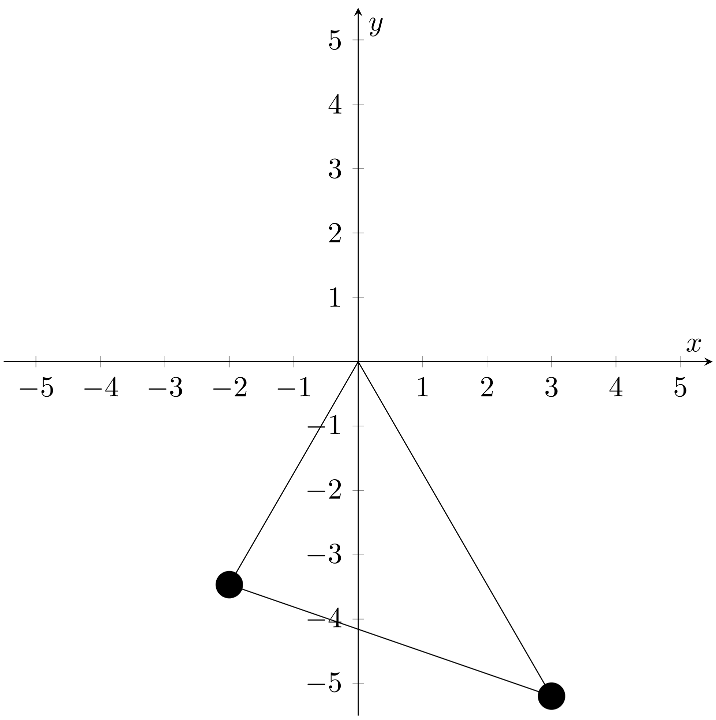
as before. We finish the problem the same way as above, obtaining a final answer of $\frac{24\sqrt{3}}{5}$.

5. Find the distance bewteen the points with polar coordinates $(4, 4\pi/3)$ and $(6, 5\pi/3)$.

Solution: The distance formula says that

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

To use this formula, we need to know the x and y coordinates of our points. Our points our given in polar coordinates, so to find the x and y coordinates we must convert them to Cartesian coordinates. A sketch can help.



The Cartesian coordiantes of the first point are

$$\left(4\cos\tfrac{4\pi}{3}, 4\sin\tfrac{4\pi}{3}\right) = (-2, -2\sqrt{3}).$$

The Cartesian coordinates of the second point are

$$\left(6\cos\tfrac{5\pi}{3}, 6\sin\tfrac{5\pi}{3}\right) = (3, -3\sqrt{3}).$$

Thus, the distance between the two points is

$$d = \sqrt{(3-(-2))^2 + \left(-3\sqrt{3} - (-2\sqrt{3})\right)^2} = \sqrt{5^2 + \sqrt{3}^2} = \sqrt{25+3} = \sqrt{28} = 2\sqrt{7}.$$

6. Find a polar equation for the curve given by the Cartesian equation $4y^2 = x$. Your answer should be of the form $r = f(\theta)$.

Solution: Using $x = r\cos\theta$ and $y = r\sin\theta$, we can rewrite this equation as

$$4r^2\sin^2\theta = r\cos\theta.$$

Diving both sides by r, we find that

$$4r\sin^2\theta = \cos\theta.$$

Isolating r, we find that

$$r = \frac{\cos\theta}{4\sin^2\theta}.$$

If we like, we can also write this answer as $r = \frac{1}{4}\cot\theta\csc\theta$.

If you are worried about division by zero when we divided by r, note that when $r = 0$ when we are at the origin. It is thus possible that when we divided by r we lost the origin as part of our curve. But in fact it turns out we did not lose anything. We can check whether or not the origin is included in our final answer $r = \frac{\cos\theta}{4\sin^2\theta}$. It is: we can still have $r = 0$ when $\cos\theta = 0$, which happens when $\theta = \frac{\pi}{2}$, for example.

7. Consider the polar curve with equation

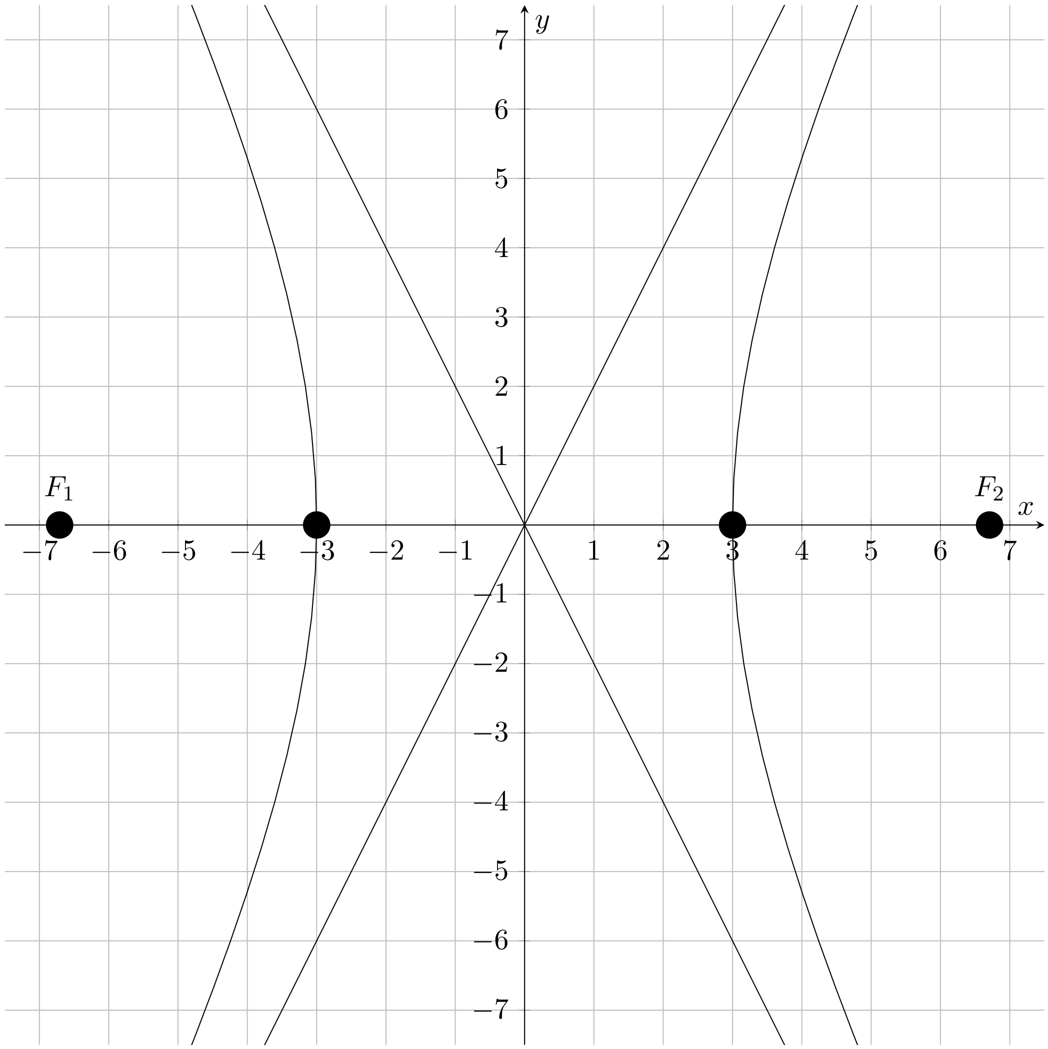
$$r = \frac{1}{\theta}, \qquad \frac{\pi}{2} \leq \theta \leq 2\pi.$$

Find the area of the region that is bounded by this curve and lies in the sector $\frac{\pi}{2} \leq \theta \leq 2\pi$.

Solution: We compute

$$\begin{aligned} A &= \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2} r^2 \, d\theta \\ &= \int_{\frac{\pi}{2}}^{2\pi} \frac{1}{2\theta^2} \, d\theta \\ &= -\frac{1}{2\theta}\bigg|_{\theta=\frac{\pi}{2}}^{2\pi} \\ &= -\left(\frac{1}{4\pi} - \frac{1}{\pi}\right) \\ &= -\frac{1-4}{4\pi} \\ &= \frac{3}{4\pi}. \end{aligned}$$

8. Sketch the hyperbola with vertices at $(\pm 3, 0)$ and asymptotes $y = \pm 2x$. Find the equation of the hyperbola. Find the foci of the hyperbola and plot them.



Solution: We plot the given vertices and the asymptotes. We draw a hyperbola through the vertices going towards the given asymptotes.

This hyperbola opens to the left and right, so its equation has the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

To check this form, we note that $y = 0$ works in this equation, but $x = 0$ does not. Because the vertices are at $(\pm 3, 0)$, we know that $a = 3$. To find the equation for the asymptotes, we replace 1 with 0 and find that

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$
$$\frac{x}{a} = \pm\frac{y}{b},$$
$$y = \pm\frac{b}{a}x.$$

Because we know that the asymptotes have equations $y = \pm 2x$, we know that $\frac{b}{a} = 2$. Since $a = 3$, we know that $b = 6$. Thus, the equation for the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{36} = 1.$$

Finally, recalling that for a hyperbola the foci are outside the vertices, we know that $c > a$, and so we use the equation $c^2 = a^2 + b^2$. We find that

$$c = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}.$$

Thus, our foci are at $(\pm 3\sqrt{5}, 0)$. Since 45 is a little smaller than 49, $\sqrt{45}$ is a little smaller than 7, which lets us plot these points.