Math 243 Final

May 12, 2020

- Write your solutions and upload them on Gradescope, just like your homework assignments. You can write your solutions on the exam pages or on separate sheets of paper, your choice.
- Only use the resources allowed on the exam honor code certification form.
- Be sure to include the exam honor code certification form with your solutions. If you are unable to print it, copy the form by hand.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.

1. Consider the curve defined by the parametric equations

$$x = 3t^2, \qquad y = 2t^3, \qquad -2 \leq t \leq 0.$$

Find the length of the curve.

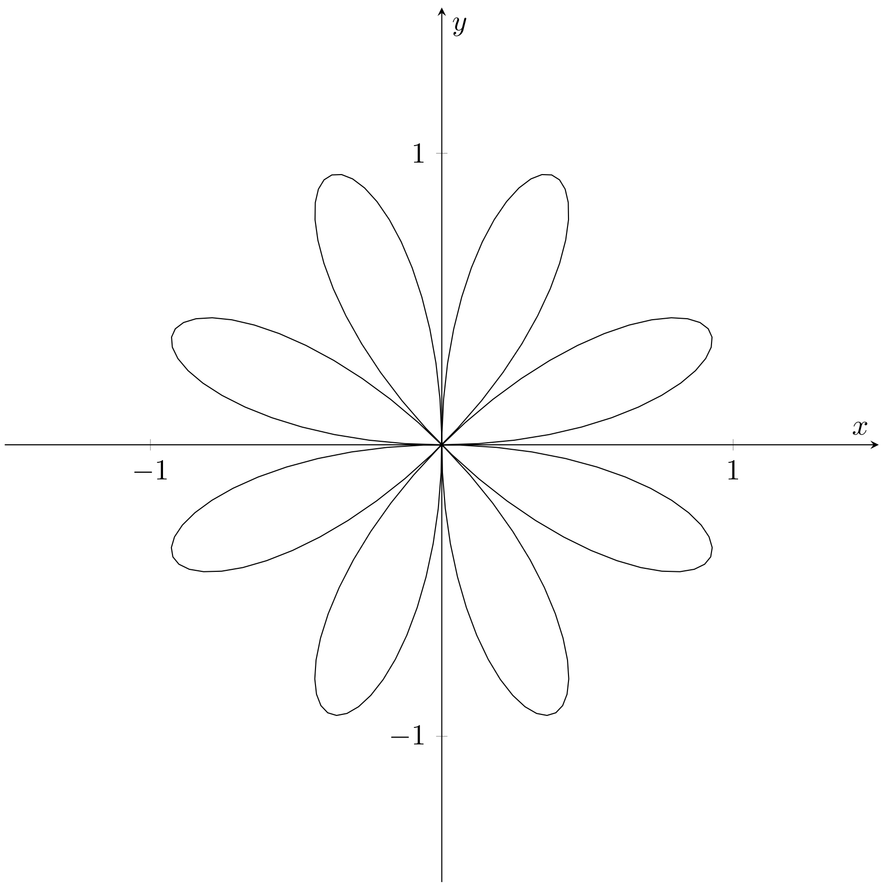
Solution: The velocity is $\langle 6t, 6t^2 \rangle = 6t\langle 1, t\rangle$, so the speed is

$$6\,|t|\,\sqrt{1+t^2} = -6t\sqrt{1+t^2}$$

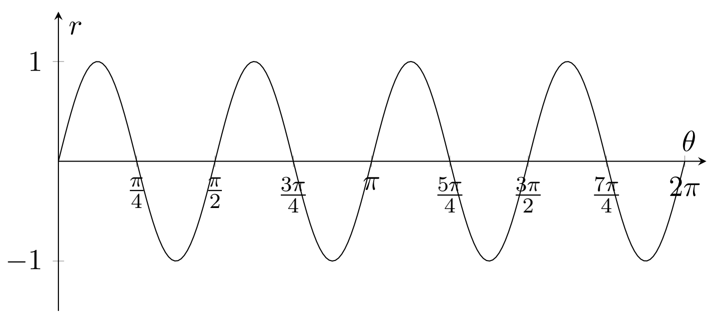
because $t \leq 0$. Since distance is speed multiplied by time,

$$\begin{aligned} L &= \int_{-2}^{0} (-6t\sqrt{1+t^2})\,dt \\ &= -3\int_{-2}^{0} \sqrt{1+t^2}\,d(1+t^2) \\ &= -2\left(1+t^2\right)^{3/2}\Big|_{-2}^{0} \\ &= -2\left(1^{3/2} - 5^{3/2}\right) \\ &= 2(5\sqrt{5}-1) = 10\sqrt{5} - 2. \end{aligned}$$

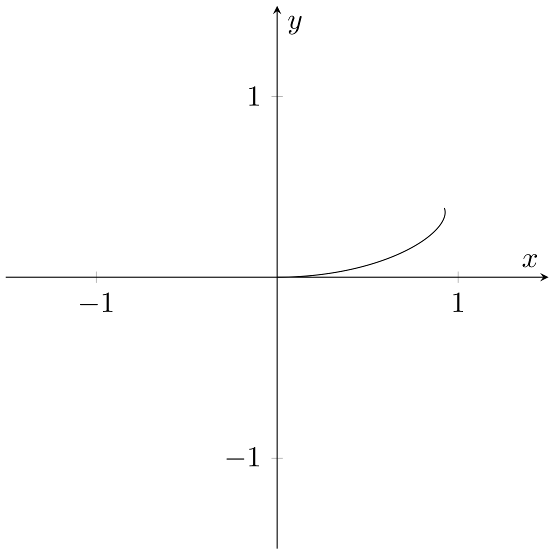
2. Sketch the polar curve $r = \sin 4\theta$.



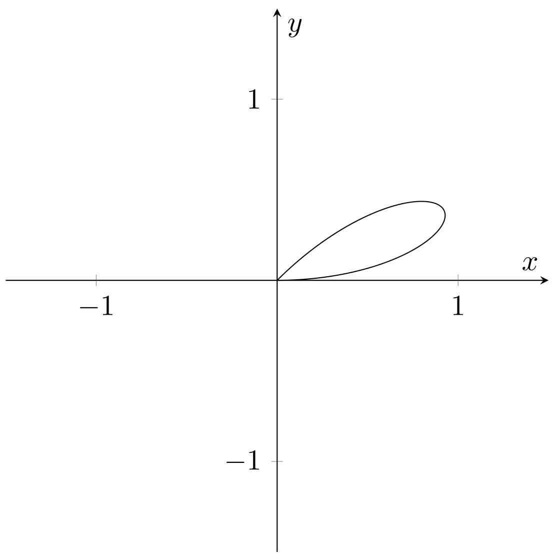
Solution: We begin with our helper Cartesian plot of $r = \sin 4\theta$.



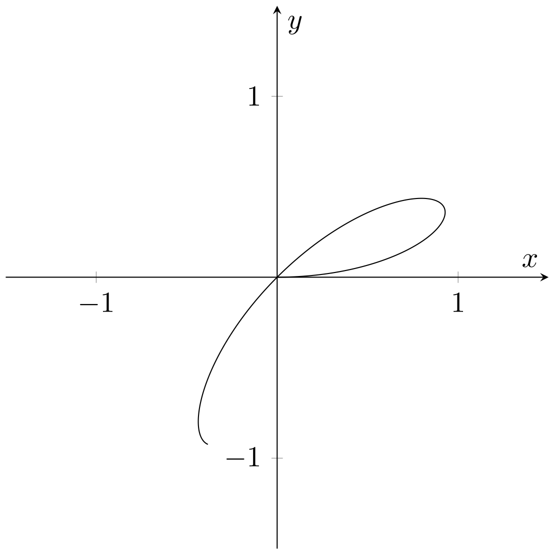
We see that as θ goes from 0 to $\frac{\pi}{8}$, r goes from 0 to 1. We draw the corresponding part of the curve in the region $0 \leq \theta \leq \frac{\pi}{8}$.



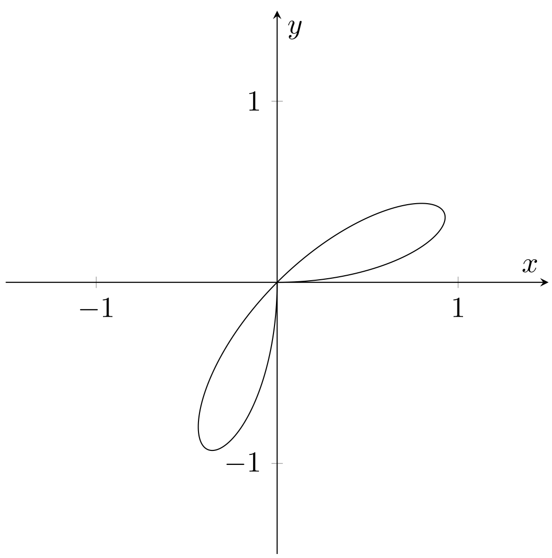
Then, as θ goes from $\frac{\pi}{8}$ to $\frac{\pi}{4}$, r goes from 1 to 0. Our plot now looks like



Then, as θ goes from $\frac{\pi}{4}$ to $\frac{3\pi}{8}$, r goes from 0 to -1. Because r is negative, we draw the next part of the curve in the region opposite the region with $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{8}$, namely the region with $\frac{3\pi}{4} \leq \theta \leq \frac{11\pi}{8}$. Our plot now looks like



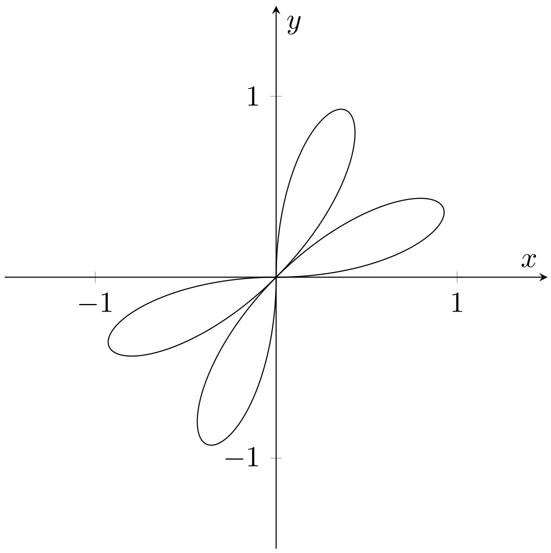
Next, as θ goes from $\frac{3\pi}{8}$ to $\frac{\pi}{2}$, r goes from -1 to 0. Because r is negative, we draw the next part of the curve in the region opposite the region with $\frac{3\pi}{8} \leq \theta \leq \frac{\pi}{2}$, namely the region with $\frac{11\pi}{8} \leq \theta \leq \frac{3\pi}{2}$. Our plot now looks like



We note that our equation does not change when we replace θ by $\theta + \pi$. Indeed,

$$\sin(4(\theta + \pi)) = \sin(4\theta + 4\pi) = \sin 4\theta.$$

Thus, our curve has 180° rotational symmetry. Rotating what we've drawn by 180°, our plot is now



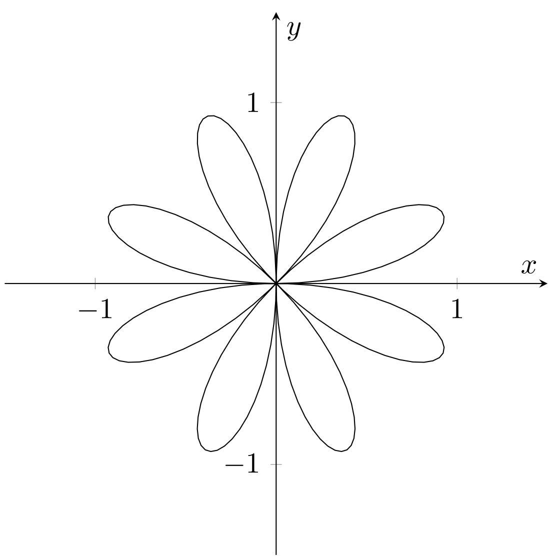
In fact, we could have saved ourselves some work by noticing that the equation does not change when we replace θ by $\theta + \frac{\pi}{2}$, so our curve is actually invariant under 90° rotations also.

Regardless, we can try replacing θ by $-\theta$. We obtain

$$\sin(4(-\theta)) = \sin(-4\theta) = -\sin 4\theta$$

because sine is odd. We see that we don't get the same exprsesion. To get the same equation, we must this replace θ by $-\theta$ and replace r by $-r$. Indeed, the equation $-r = -\sin 4\theta$ is the same as the equation $r = \sin 4\theta$.

Therefore, our curve stays the same if we reflect it across the x-axis and then rotate it by 180°. We do so, obtaining our answer.



3. (a) Find the distance between a point (x, y, z) and the y-axis.

> **Solution:** The closest point on the y-axis is $(0, y, 0)$, so, by the distance formula, the distance is $\sqrt{x^2 + z^2}$.

(b) Find the distance between a point (x, y, z) and the xz-plane.

> **Solution:** The closest point on the xz-plane is $(x, 0, z)$, so the distance is $|y|$.

(c) Consider all of the points (x, y, z) for which its distance to the y-axis is three times its distance to the xz-plane. Write down an equation that describes these points.

> **Solution:** Given all of the above, the equation is
>
> $$\sqrt{x^2 + z^2} = 3\,|y|\,,$$
>
> which we can simplify as
>
> $$x^2 + z^2 = 9y^2.$$

(d) Identify the surface.

> **Solution:** Either from the description or from the equation, we see that this surface is a cone.

4. Consider the curve described by the parametric equations

$$\mathbf{r}(t) = e^{-t}\langle 1, \sin t, \cos t\rangle.$$

You may find it useful to know that $|\mathbf{r}'(t)| = \sqrt{3}e^{-t}$.

(a) Reparametrize this curve with respect to arc length.

Solution: Speed is the change in distance over time, so

$$\begin{aligned}\frac{ds}{dt} &= \left|\mathbf{r}'(t)\right| \\ s &= \int \sqrt{3}e^{-t}\,dt = -\sqrt{3}e^{-t} + C.\end{aligned}$$

Choosing $C = 0$ for simplicity, we solve for t, finding that

$$t = -\ln\left(-\frac{s}{\sqrt{3}}\right).$$

Then, we rewrite $\mathbf{r}$ in terms of s. We have

$$\begin{aligned}\mathbf{r} &= -\frac{s}{\sqrt{3}}\left\langle 1, \sin\left(-\ln\left(-\frac{s}{\sqrt{3}}\right)\right), \cos\left(-\ln\left(-\frac{s}{\sqrt{3}}\right)\right)\right\rangle \\ &= -\frac{s}{\sqrt{3}}\left\langle 1, -\sin\ln\left(-s/\sqrt{3}\right), \cos\ln\left(-s/\sqrt{3}\right)\right\rangle.\end{aligned}$$

(b) Find two points on the curve such that the arc length between them is three units.

Solution: Since s represents distance traveled, we want two values of s that are 3 apart. A natural choice is $s = 0$ and $s = 3$, but they are not valid values for our formula because we can't take the logarithm of a negative number.

So, instead, let's pick $s = -1$ and $s = -4$. Plugging in $s = -1$, we obtain

$$\mathbf{r} = \frac{1}{\sqrt{3}}\left\langle 1, -\sin\ln\left(1/\sqrt{3}\right), \cos\ln\left(1/\sqrt{3}\right)\right\rangle.$$

Plugging in $s = -4$, we obtain

$$\mathbf{r} = \frac{4}{\sqrt{3}}\left\langle 1, -\sin\ln\left(4/\sqrt{3}\right), \cos\ln\left(4/\sqrt{3}\right)\right\rangle.$$

So, one possible answer to this question is

$$\begin{aligned}&\left(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\sin\ln\left(1/\sqrt{3}\right), \tfrac{1}{\sqrt{3}}\cos\ln\left(1/\sqrt{3}\right)\right), \\ &\left(\tfrac{4}{\sqrt{3}}, -\tfrac{4}{\sqrt{3}}\sin\ln\left(4/\sqrt{3}\right), \tfrac{4}{\sqrt{3}}\cos\ln\left(4/\sqrt{3}\right)\right).\end{aligned}$$

5. Consider a particle with position

$$\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^4 \rangle.$$

Find the velocity, acceleration, and speed of the particle. Be sure to simplify as much as possible.

Solution: We compute

$$\begin{aligned}
\mathbf{v}(t) &= \langle 2t + 1, 2t - 1, 4t^3 \rangle, \\
\mathbf{a}(t) &= \langle 2, 2, 12t^2 \rangle, \\
v(t) &= \sqrt{(2t+1)^2 + (2t-1)^2 + (4t^3)^2} = \sqrt{16t^6 + 8t^2 + 2}.
\end{aligned}$$

The radicand is not a perfect square, so we cannot simplify further.

6. Let $f(x, y, z) = x^2 + 4y^2 + 9z^2$.

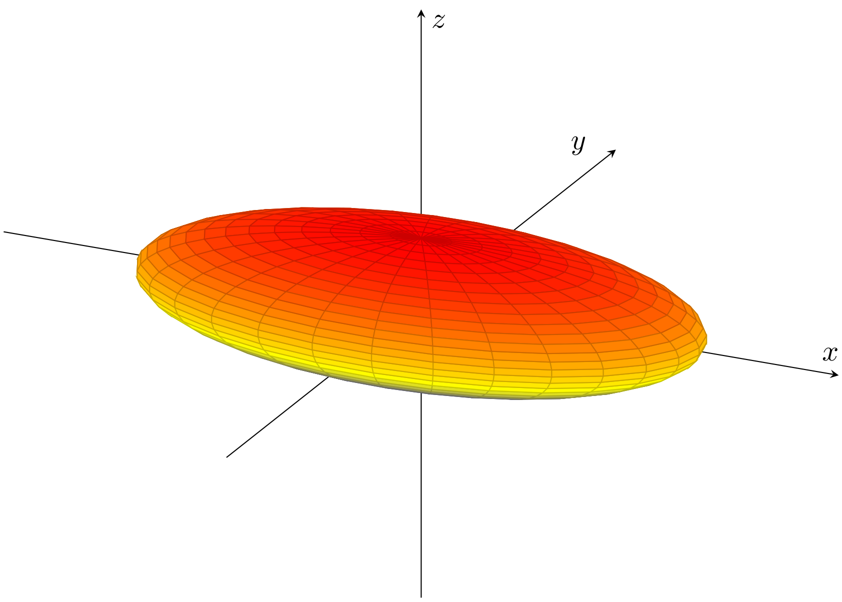
   (a) Describe the level surfaces of f.

   Solution: The level surfaces of f are defined by the equations

   $$k = x^2 + 4y^2 + 9z^2.$$

   When $k > 0$, this is the equation of an ellipsoid. When $k = 0$, this is the equation of a point at the origin. When $k < 0$, there are no solutions to this equation.

   (b) Draw one of the level surfaces of f.



   Solution: We draw an ellipsoid. We can set $k = 1$ for example. To get the dimensions right, we find the intercepts. Setting the variables to zero two at a time, we find that the x-intercepts are at $x = \pm 1$, the y-intercepts are at $y = \pm\frac{1}{2}$, and the z-interceppts are at $z = \pm\frac{1}{3}$. We draw the ellipsoid appropriately.

7. Compute the limit

$$\lim_{(x,y)\to(2,3)} (x^2y^3 - 4y^2).$$

Solution: The function $x^2y^3-4y^2$ is continuous, so we can find the limit by simply plugging in $x = 2$ and $y = 3$ to get

$$2^2 3^3 - 4 \cdot 3^2 = 4 \cdot 27 - 4 \cdot 9 = 4 \cdot 18 = 72.$$

8. Let $f(x, y, z) = e^{xyz}$. Compute f_{xyz}.

Solution: We compute

$$\begin{aligned} f_x &= yze^{xyz}, \\ f_{xy} &= ze^{xyz} + (yz)(xz)e^{xyz} = z(1 + xyz)e^{xyz}, \\ f_{xyz} &= (1 + xyz)e^{xyz} + z(xy)e^{xyz} + z(1 + xyz)(xy)e^{xyz} \\ &= (1 + xyz + xyz + xyz + x^2y^2z^2)e^{xyz} \\ &= (1 + 3xyz + x^2y^2z^2)e^{xyz}. \end{aligned}$$

The question is symmetric in x, y, and z, so our answer should be as well. We can check our work by noticing that our answer is indeed symmetric in x, y, and z.

9. (a) Write down the volume V of a cylinder of radius r and height h.

Solution: The area of a circle of radius r is πr^2, so the volume of a cylinder with radius r and height h is
$$V = \pi r^2 h.$$

(b) Compute dV.

Solution: Taking partial derivatives, we compute that
$$dV = 2\pi r h\, dr + \pi r^2\, dh$$

(c) Use part (b) to estimate the amount of aluminum in a standard US soda can with radius 3 cm, height 12 cm, and thickness 0.01 cm. Include units in your answer.

Solution: The difference between the volume of the outside of the can and the volume of the inside of the can is the volume taken up by the aluminum. Thus, we'd like to compute Δv. From part (b), we know that
$$\Delta V \approx 2\pi r h\, \Delta r + \pi r^2\, \Delta h.$$

We are given that $r = 3\,\text{cm}$ and $h = 12\,\text{cm}$. The thickness tells us that the radius of the outside of the can is 0.01 cm larger than the radius of the inside of the can. Remembering that the can has both a top and a bottom, the height of the outside of the can is $0.01\,\text{cm} + 0.01\,\text{cm} = 0.02\,\text{cm}$ more than the height of the inside of the can. Substituting these values for Δr and Δh, respectively, we find that

$$\begin{aligned}\Delta V &\approx 2\pi(3\,\text{cm})(12\,\text{cm})(0.01\,\text{cm}) + \pi(3\,\text{cm})^2(0.02\,\text{cm}) \\ &= \pi(0.72\,\text{cm}^3 + 0.18\,\text{cm}^3) = \pi(0.9\,\text{cm}^3) \approx 3\,\text{cm}^3.\end{aligned}$$

Thus, the can is made up of about three cubic centimeters of aluminum.

We can check our answer by computing the surface area of the can and multiplying by the thickness. The bottom of the can has area πr^2. The top of the can has area πr^2. The circle has circumference $2\pi r$, so the sides of the can have area $2\pi r h$. Thus, the surface area of the can is $2\pi r^2 + 2\pi r h$. Multiplying by the thickness 0.01 cm, we would obtain the same computation as before.

10. Let

$$z = f(x, y), \qquad x = r^2 + s^2, \qquad y = r^2 - s^2.$$

(a) Compute $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ in terms of r, s, f_x and f_y.

Solution: We compute

$$\frac{\partial x}{\partial r} = 2r, \qquad \frac{\partial y}{\partial r} = 2r,$$
$$\frac{\partial x}{\partial s} = 2s, \qquad \frac{\partial y}{\partial s} = -2s.$$

Then, by the chain rule

$$\frac{\partial z}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} = 2r f_x + 2r f_y,$$
$$\frac{\partial z}{\partial s} = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} = 2s f_x - 2s f_y.$$

(b) Compute $\frac{\partial^2 z}{\partial r \partial s}$ in terms of r, s, and the partial derivatives of f.

Solution: We compute $\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial s}\right)$. As in Example 7 in section 14.5, we use the product rule. This problem is easier than Example 7 because s doesn't depend on r. We find that

$$\frac{\partial^2 z}{\partial r \partial s} = 2s \frac{\partial f_x}{\partial r} - 2s \frac{\partial f_y}{\partial r}.$$

To compute $\frac{\partial f_x}{\partial r}$ and $\frac{\partial f_y}{\partial r}$, we use the chain rule.

$$\frac{\partial f_x}{\partial r} = \frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r} = 2r f_{xx} + 2r f_{xy},$$
$$\frac{\partial f_y}{\partial r} = \frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r} = 2r f_{yx} + 2r f_{yy}.$$

Plugging that in and simplifying using $f_{xy} = f_{yx}$, we find that

$$\begin{aligned} \frac{\partial^2 z}{\partial r \partial s} &= 2s(2r f_{xx} + 2r f_{xy}) - 2s(2r f_{yx} + 2r f_{yy}) \\ &= 4rs(f_{xx} - f_{yy}). \end{aligned}$$

One way to check our work would be to compute the second derivative in the other order $\frac{\partial^2 z}{\partial s \partial r}$.

11. Let $f(x, y, z) = x^2y + yz$. Compute the directional derivative of f at $(1, 2, 3)$ in the direction $\mathbf{v} = \langle 2, 1, -2\rangle$.

Solution: We compute

$$\nabla f = \langle 2xy, x^2 + z, y\rangle.$$

At $(x, y, z) = (1, 2, 3)$, we have

$$\nabla f = \langle 4, 1 + 3, 2\rangle = \langle 4, 4, 2\rangle.$$

The have

$$|\mathbf{v}| = \sqrt{4 + 1 + 4} = 3.$$

Thus, the unit vector $\mathbf{u}$ in the direction $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{3}\mathbf{v} = \frac{1}{3}\langle 2, 1, -2\rangle.$$

Therefore, the directional derivative is

$$D_{\mathbf{u}}f = \mathbf{u} \cdot \nabla f = \frac{1}{3}\langle 2, 1, -2\rangle \cdot \langle 4, 4, 2\rangle = \frac{1}{3}(8 + 4 - 4) = \frac{8}{3}.$$

12. Consider the plane $x + y + z = 1$.

(a) Write down a formula in terms of x and y, but not z, for the distance squared between a point on the plane and the point $(1, 0, -2)$.

Solution: For a point (x, y, z), the distance to $(1, 0, -2)$ is

$$\sqrt{(x-1)^2 + y^2 + (z+2)^2}.$$

For a point on the plane, $z = 1 - x - y$. Plugging that in, and squaring, we find that the distance squared to $(1, 0, -2)$ is

$$(x-1)^2 + y^2 + (3 - x - y)^2.$$

(b) Use part (a) and the techniques of Section 14.7 to find the smallest possible distance between a point on the plane and the point $(1, 0, -2)$.

Solution: Setting $f(x, y) = (x-1)^2 + y^2 + (3-x-y)^2$, we seek to minimize f. The minimum must occur at a critical point, so we compute using the chain rule that

$$\begin{aligned} f_x &= 2(x-1) + 2(3-x-y)(-1) = 2(x-1-3+x+y) = 2(2x+y-4), \\ f_y &= 2y + 2(3-x-y)(-1) = 2(y-3+x+y) = 2(x+2y-3). \end{aligned}$$

To find a critical point where $\nabla f = \mathbf{0}$, we need to solve the system

$$\begin{aligned} 2x + y &= 4, \\ x + 2y &= 3. \end{aligned}$$

By multiplying the first equation by 2 and then subtracting the second equation, we find that $3x = 5$, so $x = \frac{5}{3}$. Likewise, multiplying the second equation by 2 and then subtracting the first equation, we find that $3y = 2$, so $y = \frac{2}{3}$. We can check that $(x, y) = \left(\frac{5}{3}, \frac{2}{3}\right)$ satisfy both equations.
At this point,

$$f(x, y) = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{4}{3}.$$

Since f represents the distance squared, the shortest distance between a point on the plane and the point $(1, 0, -2)$ is $\sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$.
We can check our work by noting that the displacement vector between $(x, y, z) = \left(\frac{5}{3}, \frac{2}{3}, -\frac{4}{3}\right)$ and $(1, 0, -2)$ is $\left\langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$, which is parallel to the normal vector $\mathbf{n} = \langle 1, 1, 1 \rangle$ for the plane, as we'd expect from the closest point on the plane.

13. Consider the function $f(x, y) = y\cos x$. Observe that

$$f_x = -y\sin x, \qquad f_y = \cos x.$$

Observe also that $(x, y) = \left(\frac{\pi}{2}, 0\right)$ is a critical point of f.

Write down the quadratic approximation to f at $\left(\frac{\pi}{2}, 0\right)$.

Solution: To compute the quadratic approximation, we need the second derivatives, so we compute

$$f_{xx} = -y\cos x, \qquad f_{xy} = -\sin x, \qquad f_{yy} = 0.$$

Evaluating at $(x, y) = \left(\frac{\pi}{2}, 0\right)$, we have that $f = 0$. Because we are givien that this point is a critical point for f, we know that $f_x = f_y = 0$. From our second derivatives, we compute that, at this point,

$$f_{xx} = 0, \qquad f_{xy} = -1, \qquad f_{yy} = 0.$$

Thus, from the formula, the quadratic approximation is

$$Q(x, y) = -\left(x - \frac{\pi}{2}\right)y.$$

We can check our work by noting that

$$Q(x, y) = -\left(x - \frac{\pi}{2}\right)y \approx -\sin\left(x - \frac{\pi}{2}\right)y = \sin\left(\frac{\pi}{2} - x\right)y = \cos(x)y = f(x, y).$$

14. Let $f(x, y, z) = x^3 + y^3 + z^3$, and consider the sphere $x^2 + y^2 + z^2 = 1$.

(a) Use the method of Lagrange multipliers to set up a system of equations for finding the extreme values of f on the sphere. You should have four equations with four unknowns.

Solution: Let $g(x, y, z) = x^2 + y^2 + z^2$. The method of Lagrange multipliers tells us to solve

$$\nabla f = \lambda \nabla g,$$
$$\langle 3x^2, 3y^2, 3z^2 \rangle = \lambda \langle 2x, 2y, 2z \rangle.$$

Along with the constraint, this gives us the system of equations

$$\begin{aligned} 3x^2 &= 2\lambda x \\ 3y^2 &= 2\lambda y \\ 3z^2 &= 2\lambda z \\ x^2 + y^2 + z^2 &= 1. \end{aligned}$$

(b) Find all solutions to this system of equations. For each solution, make sure to give the value of the Lagrange multiplier λ, not just the values of x, y, and z.

If you choose to use the notation $\pm$, make sure to clarify what you mean. $(\pm 1, \pm 1)$ can mean just $(1, 1)$ and $(-1, -1)$, or it can mean all four possibilities: $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. If your solution leaves it ambiguous, you won't get credit.

Hint: Don't divide by zero. If you do, you'll find some of the solutions, but not all of them.

Solution: From the first equation, either $x = 0$ or $x = \frac{2}{3}\lambda$. Likewise, either $y = 0$ or $y = \frac{2}{3}\lambda$, and either $z = 0$ or $z = \frac{2}{3}\lambda$.

We can't have all three variables be zero because $x^2 + y^2 + z^2 = 1$, which fails if $x = y = z = 0$.

If two of the variables are zero, then, by symmetry, we can consider the case where $y = z = 0$ and $x \neq 0$. In that case, $x^2 + y^2 + z^2 = 1$ implies that $x = \pm 1$. Since $x = \frac{2}{3}\lambda$, we have $\lambda = \pm\frac{3}{2}$. By symmetry, we have corresponding solutions for the other variables.

Thus, with $\lambda = \frac{3}{2}$, we have the solutions

$$(1, 0, 0), (0, 1, 0), (0, 0, 1),$$

and with $\lambda = -\frac{3}{2}$, we have the solutions

$$(-1, 0, 0), (0, -1, 0), (0, 0, -1).$$

If one of the variables is zero, then, by, by symmetry, we can consider the case where $z = 0$ and x and y are nonzero. In that case, substituting $x = \frac{2}{3}\lambda$ and $y = \frac{2}{3}\lambda$ into

$x^2 + y^2 = 1$, we obtain

$$2\left(\frac{2}{3}\lambda\right)^2 = 1,$$
$$\lambda^2 = \frac{3^2}{2^3},$$
$$\lambda = \pm\frac{3}{2\sqrt{2}},$$

so $x = y = \pm\frac{1}{\sqrt{2}}$, where the signs match the sign of λ. By symmetry, we have corresponding solutions for the other variables.
Thus, with $\lambda = \pm\frac{3}{2\sqrt{2}}$, we have the solutions

$$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right),$$

where the signs match the sign of λ.
Finally, if none of the variables are zero, then substituting $x = y = z = \frac{2}{3}\lambda$ into $x^2 + y^2 + z^2 = 1$, we obtain

$$3\left(\frac{2}{3}\lambda\right)^2 = 1,$$
$$\lambda = \pm\frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2},$$

so $x = y = z = \pm\frac{1}{\sqrt{3}}$.
Thus, with $\lambda = \pm\frac{3}{2\sqrt{3}}$, we have the solution

$$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right),$$

where the signs match the sign of λ.

(c) Don't finish the problem. Normally, you'd plug in all of those points into f and figure out which value is the biggest and the smallest. I'm putting this part here because you're probably expecting to do more work to finish the problem, but I want you to go back and check your work on the other problems instead. Don't do any work for this part. No, seriously, it's worth zero points. No extra credit or anything. Don't do it.