Math 243 Final

May 12, 2020

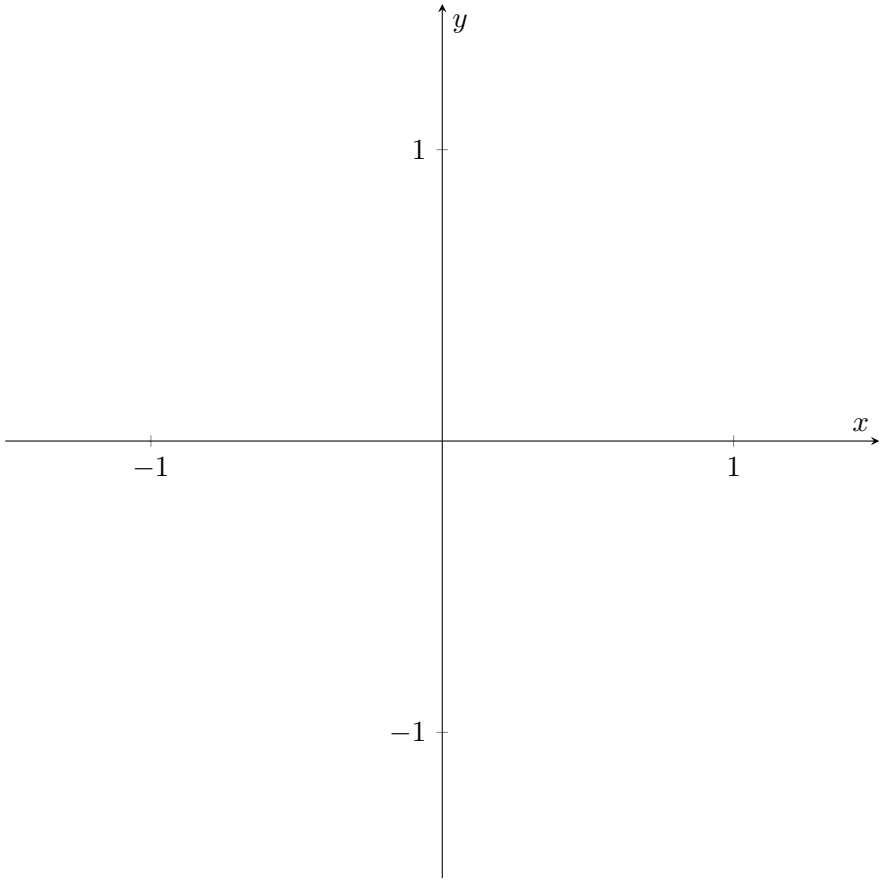
- Write your solutions and upload them on Gradescope, just like your homework assignments. You can write your solutions on the exam pages or on separate sheets of paper, your choice.
- Only use the resources allowed on the exam honor code certification form.
- Be sure to include the exam honor code certification form with your solutions. If you are unable to print it, copy the form by hand.
- Show enough work that your solution would convince a skeptical peer that your answer is correct.
- The questions are ordered by topic, not by difficulty.
- Each question is worth the same number of points.

1. Consider the curve defined by the parametric equations

$$x = 3t^2, \qquad y = 2t^3, \qquad -2 \leq t \leq 0.$$

Find the length of the curve.

2. Sketch the polar curve $r = \sin 4\theta$.



3. (a) Find the distance between a point (x, y, z) and the y-axis.

   (b) Find the distance between a point (x, y, z) and the xz-plane.

   (c) Consider all of the points (x, y, z) for which its distance to the y-axis is three times its distance to the xz-plane. Write down an equation that describes these points.

   (d) Identify the surface.

4. Consider the curve described by the parametric equations

$$\mathbf{r}(t) = e^{-t}\langle 1, \sin t, \cos t\rangle.$$

You may find it useful to know that $|\mathbf{r}'(t)| = \sqrt{3}e^{-t}$.

(a) Reparametrize this curve with respect to arc length.

(b) Find two points on the curve such that the arc length between them is three units.

5. Consider a particle with position

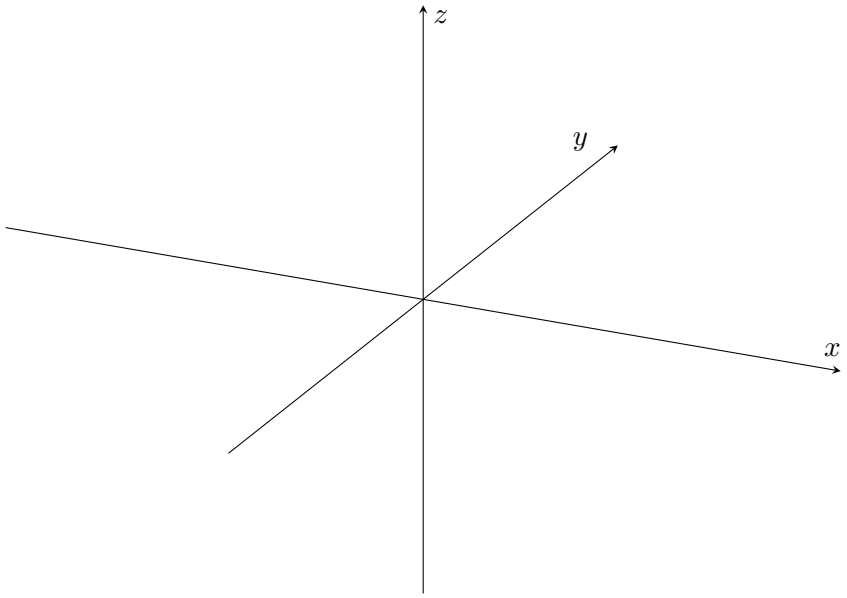
$$\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^4 \rangle.$$

Find the velocity, acceleration, and speed of the particle. Be sure to simplify as much as possible.

6. Let $f(x, y, z) = x^2 + 4y^2 + 9z^2$.

   (a) Describe the level surfaces of f.

   (b) Draw one of the level surfaces of f.



7. Compute the limit

$$\lim_{(x,y)\to(2,3)} (x^2y^3 - 4y^2).$$

8. Let $f(x, y, z) = e^{xyz}$. Compute f_{xyz}.

9. (a) Write down the volume V of a cylinder of radius r and height h.

(b) Compute dV.

(c) Use part (b) to estimate the amount of aluminum in a standard US soda can with radius 3 cm, height 12 cm, and thickness 0.01 cm. Include units in your answer.

10. Let

$$z = f(x, y), \qquad x = r^2 + s^2, \qquad y = r^2 - s^2.$$

(a) Compute $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ in terms of r, s, f_x and f_y.

(b) Compute $\frac{\partial^2 z}{\partial r \partial s}$ in terms of r, s, and the partial derivatives of f.

11. Let $f(x, y, z) = x^2y + yz$. Compute the directional derivative of f at $(1, 2, 3)$ in the direction $\mathbf{v} = \langle 2, 1, -2 \rangle$.

12. Consider the plane $x + y + z = 1$.

    (a) Write down a formula in terms of x and y, but not z, for the distance squared between a point on the plane and the point $(1, 0, -2)$.

    (b) Use part (a) and the techniques of Section 14.7 to find the smallest possible distance between a point on the plane and the point $(1, 0, -2)$.

13. Consider the function $f(x, y) = y \cos x$. Observe that

$$f_x = -y \sin x, \qquad f_y = \cos x.$$

Observe also that $(x, y) = \left(\frac{\pi}{2}, 0\right)$ is a critical point of f.

Write down the quadratic approximation to f at $\left(\frac{\pi}{2}, 0\right)$.

14. Let $f(x, y, z) = x^3 + y^3 + z^3$, and consider the sphere $x^2 + y^2 + z^2 = 1$.

    (a) Use the method of Lagrange multipliers to set up a system of equations for finding the extreme values of f on the sphere. You should have four equations with four unknowns.

    (b) Find all solutions to this system of equations. For each solution, make sure to give the value of the Lagrange multiplier λ, not just the values of x, y, and z.

    If you choose to use the notation $\pm$, make sure to clarify what you mean. $(\pm 1, \pm 1)$ can mean just $(1, 1)$ and $(-1, -1)$, or it can mean all four possibilities: $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. If your solution leaves it ambiguous, you won't get credit.

    Hint: Don't divide by zero. If you do, you'll find some of the solutions, but not all of them.

    (c) Don't finish the problem. Normally, you'd plug in all of those points into f and figure out which value is the biggest and the smallest. I'm putting this part here because you're probably expecting to do more work to finish the problem, but I want you to go back and check your work on the other problems instead. Don't do any work for this part. No, seriously, it's worth zero points. No extra credit or anything. Don't do it.