Math 4121 Midterm 2

March 27, 2019

Name: ______________________________

- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- Give and use definitions from the book or from class.
- You may use any results you remember from the book or from class as long as they are more basic than the result you're asked to prove.

1. (a) (5 points) State the Monotone Convergence Theorem.

Solution: Let $f_n \in L$. Assume that $f_1 \leq f_2 \leq \cdots$ and that the integrals $\int f_n$ are bounded above. Then the sequence f_n converges almost everywhere to a function $f \in L$, and $\int f_n \to \int f$.

(b) (5 points) State the Dominated Convergence Theorem.

Solution: Let $f_n \in L$. Assume that the sequence f_n converges almost everywhere to a function f, and that there exists a function $g \in L$ such that, for all n, $|f_n| \leq g$ a.e.. Then $f \in L$ and $\int f_n \to \int f$.

(c) (5 points) State Fatou's Lemma (either version in the textbook).

Solution: Let $f_n \in L$. Assume that $f_n \geq 0$, that the sequence f_n converges almost everywhere to a function f, and that the integrals $\int f_n$ are bounded above. Then $f \in L$ and

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

2. (20 points) Give an example of functions f_n and f defined on $[a, b]$ such that

   - f_n and f are in L and
   - $f_n \to f$ a.e., but
   - $\int f_n \not\to \int f$.

   Make sure to at least briefly explain why your example satisfies each condition.

Solution: One example we saw in class is

$$f_n = n\chi_{(0,1/n)}$$

and $f = 0$, defined on $[0, 1]$.

- f_n and f are step functions, so they are in L.
- For $x \neq 0$, we have that $f_n(x) = 0$ for $n \geq \frac{1}{x}$. Thus $f_n(x) \to 0 = f(x)$. This suffices to show that $f_n \to f$ a.e., though we can also see that when $x = 0$ we have $f_n(x) = 0 \to 0 = f(0)$.
- $\int f_n = n(1/n) = 1$, whereas $\int f = 0$, so $\int f_n \not\to \int f$.

We can easily modify this example to work on any interval $[a, b]$ by applying a translation and a dilation, resulting in $g_n(x) = f_n(\frac{x-a}{b-a})$ and similarly for g. This operation multiplies the integrals by a factor of $b - a$.

3. Let f_n be a sequence of measurable functions on $[a, b]$. Assume that for almost every x we have that $\sup\{f_n(x) \mid n \in \mathbb{N}\}$ exists, and define $f = \sup_n f_n$.

(a) (10 points) Show that f is measurable.

> **Solution:** Let $g_n = \max\{f_1, \ldots, f_n\}$. We showed in class that the maximum of measurable functions is measurable, so g_n is measurable. Moreover, $f = \lim_{n\to\infty} g_n$ almost everywhere. Indeed, $\{g_n(x)\}_{n=1}^{\infty}$ is an increasing sequence, and so if $\sup_n f_n(x)$ exists we have that $\lim_{n\to\infty} g_n(x) = \sup_n g_n(x) = \sup_n f_n(x)$. We showed in class that the limit of measurable functions is measurable, so f is measurable.

(b) (10 points) Assume furthermore that there exists an integrable function g such that for all n we have that $|f_n| \leq g$ almost everywhere. Show that f is integrable.

> **Solution:** We have that $-g \leq f_n \leq g$ almost everywhere. Taking the supremum, we see that $-g \leq f \leq g$ almost everywhere. In other words, $|f| \leq g$ almost everywhere. We showed in class that, in this situation, if f is measurable and g is integrable, then f is integrable as well.
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> A slightly longer alternative solution would note that $|f_n| \leq g$, along with the fact that f_n is measurable and g is integrable, implies that f_n is integrable. Using the notation from the solution of part (a), we then see that g_n is integrable because the maximum of integrable functions is integrable. We also have that $|g_n| \leq g$, so we can apply the Dominated Convergence Theorem to g_n to find that the limit f is integrable.

(c) (10 points) Give an example of functions f_n such that each f_n is integrable and $f = \sup_n f_n$ exists almost everywhere, but f is not integrable.

Make sure to at least briefly explain why your example satisfies each condition.

> **Solution:** One approach is to start with an f that we showed is not integrable, such as $f\colon [0, 1] \to \mathbb{R}^*$ defined by $f(x) = \frac{1}{x}$, which we showed is not integrable in the homework. Then we can pick a sequence f_n that approximates it. Set $f_n = f\chi_{[1/n,1]}$, so f_n is zero for $0 \leq x < \frac{1}{n}$ and equal to f otherwise. We see that $f_n(x) \leq n$ for all x. Thus, each f_n is bounded and continuous almost everywhere, so it is Riemann integrable and hence integrable. But $\sup_n f_n = f$ almost everywhere. Indeed, for $x \neq 0$, we have that $f_n(x) = 0$ when $n < \frac{1}{x}$ and $f_n(x) = f(x)$ for $n \geq \frac{1}{x}$, so $\sup_n f_n(x) = f(x)$. We have shown that f_n is integrable and that $f = \sup_n f_n$ exists almost everywhere but is not integrable.
>
> Another approach is to consider the example $f_n = n^2\chi_{(0,\frac{1}{n})}$ from the textbook. In this case, f_n is a step function and so clearly integrable. We can also see that $\sup_n f_n(x)$ exists for all x. For $x = 0$ we have $f_n(x) = 0$, and for each $x > 0$, we have that $x \in (0, \frac{1}{n})$ if and only if $n < \frac{1}{x}$. Thus, $f_n(x) = n^2$ if $n < \frac{1}{x}$ and zero otherwise, so we conclude that $\sup_n f_n(x) = n^2$ where n^2 is the largest integrer smaller than $\frac{1}{x}$. Thus, indeed, $f = \sup_n f_n$ exists everywhere. But if f were integrable, since $f \geq f_n$ for all n we would have $\int f \geq \int f_n = n^2(\frac{1}{n}) = n$ for all n, which is impossible. Thus, f is not integrable.

The second example would also work with $f_n = n\chi_{(0,\frac{1}{n})}$, but the proof that f is not integrable would be more complex.
Any of these approaches can be done on a general interval $[a, b]$ by scaling and dilating as in the solution to the previous problem.

4. In this question, you will prove the following proposition in two different ways.

Proposition (Continuity of measure from below)**.** *Let E_n be a sequence of measurable subsets of $[a, b]$, nested so that*

$$E_1 \subseteq E_2 \subseteq \cdots .$$

Let $E = \bigcup_{n=1}^{\infty} E_n$. Then

$$m(E) = \lim_{n\to\infty} m(E_n).$$

(a) i. (5 points) State without proof the property of measure called *countable additivity.*

Solution: If F_n are disjoint measurable sets and $F = \bigsqcup_{n=1}^{\infty} F_n$, then

$$m(F) = \sum_{n=1}^{\infty} m(F_n).$$

ii. (15 points) Prove the above claim using countable additivity.

Solution: Let $F_n = E_n \setminus E_{n-1}$, where we set $E_0 = \emptyset$. Then the F_n are measurable because measurable sets are closed under intersections and complements. They are disjoint because if $n < m$, then if $x \in F_n$, then $x \in E_n \subseteq E_{m-1}$, and so $x \notin F_m$.

Let $F = \bigsqcup_{n=1}^{\infty} F_n$. I claim that $F = E$. Indeed, if $x \in F$, then for some n we have $x \in F_n \subseteq E_n \subseteq E$. Conversely, if $x \in E$, then $x \in E_n$ for some n. Choose the smallest such n, so $x \notin E_{n-1}$, and so $x \in F_n \subseteq F$.

By countable additivity, $m(E) = m(F) = \sum_{n=1}^{\infty} m(F_n)$. Meanwhile, since $E_n = E_{n-1} \sqcup F_n$, by finite additivity we know that $m(E_n) = m(E_{n-1}) + m(F_n)$. Thus,

$$m(E) = \sum_{n=1}^{\infty} (m(E_n) - m(E_{n-1})) = \lim_{N\to\infty} \sum_{n=1}^{N} (m(E_n) - m(E_{n-1})) = \lim_{N\to\infty} m(E_N)$$

because the sum telescopes.

(b) (15 points) Prove the above claim using the Monotone Convergence Theorem.

Solution: Let $f_n = \chi_{E_n}$. The measurability of the E_n implies that $f_n \in L$, and the nestedness of the E_n implies that $f_1 \le f_2 \le \cdots$. Because $f_n \le 1$, the integrals $\int f_n$ are bounded above by $\int 1 = b - a$. Therefore, the monotone convergence theorem implies that f_n converges almost everywhere to a function $f \in L$ and $\int f_n \to \int f$.

I claim that $f_n \to \chi_E$. Indeed, if $x \in E$, then $x \in E_N$ for some N, and by nestedness we have that $x \in E_n$ for all $n \ge N$. Thus $f_n(x) = 1$ for all $n \ge N$, so $f_n(x) \to 1 = \chi_E(x)$. On the other hand, if $x \notin E$, then $x \notin E_n$ for all n, so $f_n(x) = 0$ for all n, so $f_n(x) \to 0 = \chi_E(x)$.

Thus $f = \chi_E$ almost everywhere, so $\int f = \int \chi_E = m(E)$. Meanwhile, $\int f_n = \int \chi_{E_n} = m(E_n)$, so we conclude that $m(E_n) \to m(E)$, as desired.